

Some Mathematical Aspects of Hollow Fibre Ultrafiltration

1. INTRODUCTION

This problem was presented to the 1986 Mathematics-in-Industry Study Group by Dr D.L. Ford, the Director for Research of Memtec Ltd. Memtec specialises in ultrafiltration equipment with uses in the separation of oil-water mixtures, the purification of water, the filtration of fruit juices and wine, the treatment of wastes, and many other industrial processes. The heart of the Memtec filtration system is a bundle of about 3000 very small hollow fibres made out of plastic foam (see figure 1a). The hollow fibres have millions of tiny holes in their walls: in cases where homogeneous and isotropic foams are used, the diameter of the small holes is typically $0.1-0.2 \mu$ ($1\mu = 1$ micron = 10^{-6} m), whilst the hollow fibres have typical inner and outer radii of 100 and 300 μ . The fibres are contained in a cartridge about 15-50 cm long and 7 cm in diameter, and the material to be cleaned or separated is circulated around the fibres at low pressures (typically 100-400 kPa above atmospheric pressure). Under these pressures, the permeate moves almost radially through the wall of the fine tubes, and then axially along the inside (or the lumen) to be gathered at the ends of the cartridge. The filtration surface is the outside of the tubes.

This process allows regular cleaning of the filtration surface. To do this, the ends of the lumen are exposed to a high pressure air pulse at typically 500-700 kPa (see figure 1b). The fluid driven ahead of this pulse expands the foam in the tube walls by up to 10% in all directions and this loosens particles stuck in the holes of the walls. Subsequently, the air breaks through when the air pressure is high enough to overcome surface tension in the tiny holes. As the viscosity of air is 55 times smaller than the viscosity of water at 20°C, the expanding air moves through the tube walls relatively quickly during decompression and expels the small particles which had blocked the tube walls. The combined effects of back pressure from the inside and the use of elastic foam tubes allows the cleaning operation to be performed many times, and the cleaning only takes a small time. The cleaning process would not be as effective if the filtration surface was the inside of the tubes as is the conventional practice (see Breslau *et al.*, 1980). To see this, consider that the 100 kPa pressure of the filtration phase is not

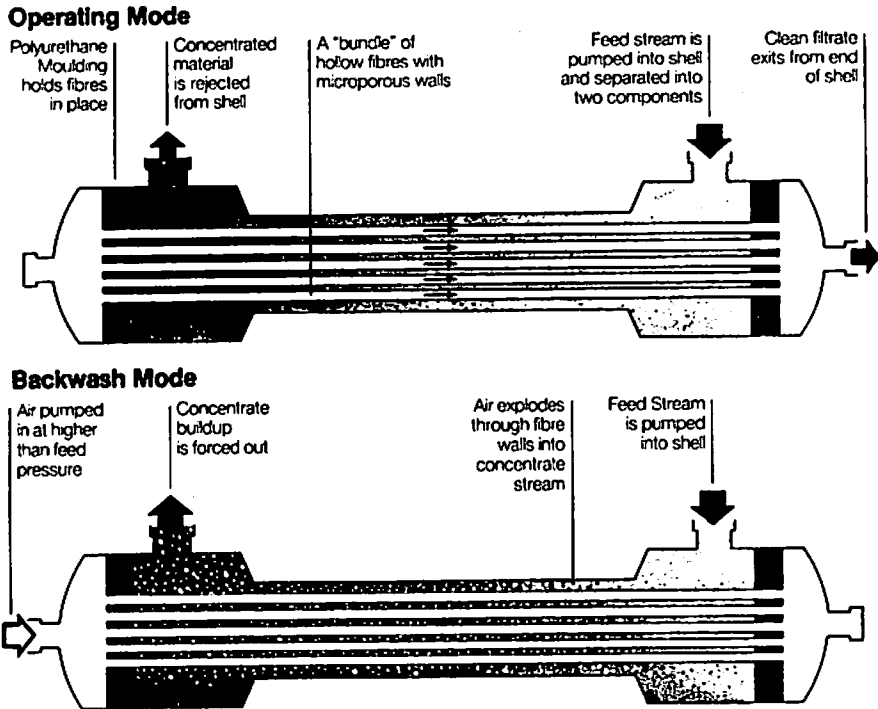


Figure 1a (top) and 1b (bottom): Illustrating the operation of the Memtec filtration cartridges.

sufficient to cause the tubes to collapse or buckle, whilst the 500-700 kPa pressure of the cleaning phase might cause thin tubes to buckle if it were applied from the outside. Also, an attempt to clean the tubes by applying pressure from the outside would contract the pores, thereby gripping dirt particles more tightly. When applied to the inside, however, the 500-700 kPa cleansing pressure is sufficient to loosen the dirt particles from the expanded pores, but not so great that it would rupture the tubes.

Memtec regards its ultrafiltration process as unique, and considers that it has a lead of between 2 and 5 years on any competitors. To preserve that competitive lead, Memtec is seeking to optimise its present products and to develop new products. For these reasons, Memtec sought mathematical assistance at the 1986 MISG.

In preliminary discussions, the following five topics were identified for

consideration at the 1986 MISG:

- (1) To derive a mathematical model of the filtration process for the purification of tap water in which there is very little dirt or particles so that the filtration surface of the tubes does not become badly fouled. The goal is to derive the total flux Q_{tot} of permeate from the cartridge as it depends on the parameters N (the number of tubes), r_i (the inner radius of the tubes), r_o (the outer radius of the tubes), A (the cross-sectional area of the cartridge), X (the half-length of the cartridge), μ (the viscosity of the permeate), k (the permeability of the tube walls), and Δp (the pressure drop between the unfiltered material surrounding the tubes and the pressure at the exit of the lumen).

The mathematical model which was derived was based on the observation that the ratio [length of tubes:wall thickness] is very large (typically about 10^3). Hence the flow of permeate through the walls is almost radial and the flow in the lumen is almost parallel. The model showed that the flux Q from an individual tube would be initially increased by increasing the half length X although, eventually, no further increase in Q is obtained by increasing X . The model provides a framework for other cases in which the tubes become blocked. The model is presented in Section 2, and some preliminary comments on optimization of the total flux Q_{tot} are presented in Section 3. This shows that it is crucial to establish values of Δp for which the tubes will collapse or fall.

- (2) The failure of the hollow fibres due to externally imposed stress can either be due to buckling of the walls, or to the stresses in the relatively thick walled tubes rising above the yield stress of the foam walls thereby causing plastic collapse. In particular, a distinction has to be made between "non-wetting" failure (in which the tube walls do not support a radially-inwards flow of permeate) and "wetting" failure (in which the presence of permeate in the tube walls provides a body force on the foam and would eventually lead to plastic collapse). The "non-wetting" case is investigated in Section 4 and the "wetting" case in Section 5.
- (3) The process by which the fibres are cleaned also needs examination. This process involves features such as: swelling of the foam due to the imposed internal pressure, the eventual passage of air through the foam walls, and the possibility that the foam walls would rupture under the imposed stresses. This cleaning process was considered to be of great

importance, but there was not sufficient time during the MISG to examine it in detail.

- (4) It was suggested by Dr Ford that the optimum design for the tube walls would be one in which the foam had a honeycomb structure with large holes about 2μ across separated by thin foam walls containing holes about 0.1μ across. The adoption of such a honeycomb foam would reduce the resistance to radial flow of the permeate, and would not compromise the strength of the walls required to avoid compressive failure during the filtration or rupture during the cleaning process. There was insufficient time at the MISG for a thorough investigation of this important topic.
- (5) Finally, a simple control problem was identified. Memtec had discovered that the total flux of permeate could be described by an empirical law (equation 6.1) which has some theoretical justification and in which two parameters have to be determined by least squares fitting of the model to flux data. The time at which the filter should be cleaned can be determined using this law and the fact that the cleaning process occupies a certain amount of time. This decision making process suffers from two practical defects: firstly, it requires an integration of data and, secondly, the flux drifts in time so that the two parameters in the model have to be updated. The goal was to suggest a control process which avoids both of these defects if possible. In fact, the control mechanism which was deduced avoids the second difficulty but still has to face the first. This topic is examined in Section 6 and it is shown that the equation which determines when to clean the filter (equation 6.3) is distribution-free and non-parametric.

2. FLUX OF PERMEATE THROUGH THE HOLLOW FIBRES

The application that is envisaged is one in which the material to be filtered is quite clean and the filtration surface takes a long time to become blocked. The foam walls of the hollow fibres are relatively thick in comparison to the inner radius and the pressure drop across the walls is typically less than 100 kPa. Under these circumstances, the walls can be regarded as being stiff and permeable. (A more detailed investigation as to when the walls are not stiff - by virtue of buckling or plastic failure - is presented in Section 4.)

The immediate goal is to find the permeate flux through the permeable foam tube walls, and to relate this to the flux of permeate through the lumen. Some of the notation has been introduced in point 1 in Section 1, and figure 2 displays the configuration and the co-ordinate system. It is noteworthy that an incomplete analysis of this problem has been given in the well-known text by Bird, Stewart and Lightfoot (1960, p.151).

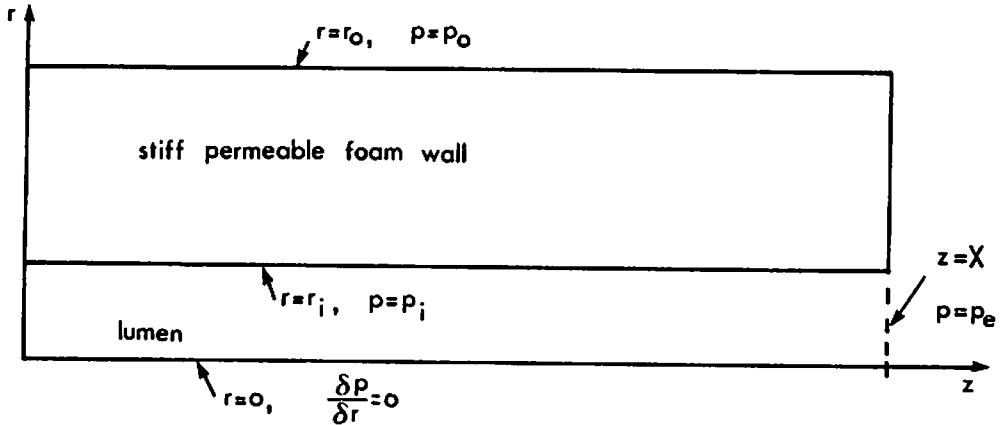


Figure 2: Definition sketch for the calculations of Section 2.

The flow speed v in the tube walls is given by Darcy's Law (see e.g. Batchelor, 1967, pp.223-224, for a crisp justification)

$$\nabla p = -\mu v/k \quad (2.1)$$

where μ is the permeate viscosity, k is a constant called the permeability, and p, v are the pressure and velocity averaged over a volume whose dimensions are large compared with the individual holes in the foam (typically 0.1μ) and small compared with the thickness of the tube walls (typically 200μ). The theoretical justification for Darcy's Law is that the flow through the medium is as if it consisted of a number of tubes of small diameter in each of which the flow is of Poiseuille type with the speed proportional to the pressure gradient and inversely proportional to the viscosity. Non-linear inertia forces are negligible provided the holes in the foam walls are suitably small.

The permeate flow in the tube walls is incompressible and almost radial in direction since the ratio [length of tubes:wall thickness] is very large. The continuity equation then gives

$$\nabla \cdot \mathbf{v} = 0$$

which implies

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{k}{\mu} \frac{\partial p}{\partial r} \right) = 0$$

and hence, in the tube walls,

$$p(r) = p_1 - C(z) \int_r^{r_0} \frac{\mu dr'}{r'k(r')} \quad (2.2a)$$

$$v_{\text{rad}}(r) = -C(z)/r. \quad (2.2b)$$

Here $C(z)$ has dimensions of

$$[\text{pressure permeability/viscosity}] = [ML^{-1}T^{-2} \cdot L^2 / ML^{-1}T^{-1}] = [L^2T^{-1}]$$

and is a slowly varying function of the axial co-ordinate z . Suppose now that the almost radial flow holds for $z > z_0$. Then the total volume flux of permeate through the inside wall of the tube up to station z is

$$Q(z) = \text{constant} + \int_{z_0}^z \int_0^{2\pi} \frac{C(z')}{r} r \, d\theta \, dz'$$

or

$$Q(z) = \text{constant} + 2\pi \int_{z_0}^z C(z') \, dz'. \quad (2.3)$$

Now consider the flow in the lumen. This flow is pressure driven and almost parallel since the ratio [inner tube radius:tube length] is very small. Locally, the pressure gradient can be regarded as constant and the non-linear inertia forces can be neglected. Thus the approximate expression for the axial velocity $u(r)$ is Poiseuille's law

$$u(r) = -\frac{1}{4\mu} \frac{dp}{dz} (r_i^2 - r^2) \quad (2.4)$$

and the local volume flux swept past station z is

$$Q(z) = \int_0^{r_i} \int_0^{2\pi} u(r) r \, dr \, d\theta$$

or

$$Q(z) = - \frac{\pi r_i^4}{8\mu} \frac{dp}{dz} . \quad (2.5a)$$

Using equation (2.2a), the local pressure gradient inside the lumen is

$$\frac{dp}{dz} = - \alpha \frac{dC}{dz}$$

where

$$\alpha = \int_{r_i}^{r_0} \frac{\mu dr}{rk(r)} . \quad (2.5b)$$

and α has the dimensions $[L \cdot ML^{-1}T^{-1}/L \cdot L^2] = [ML^{-3}T^{-1}]$. Thus equation (2.5a) becomes

$$Q(z) = \frac{\alpha \pi r_i^4}{8\mu} \frac{dC}{dz} , \quad (2.5c)$$

and, if it is assumed that this expression for the flux is true for all z , the two flux expressions (2.3) and (2.5c) give that

$$\text{constant} + 2\pi \int_{z_0}^z C(z') dz' = \frac{\alpha \pi r_i^4}{8\mu} \frac{dC}{dz} . \quad (2.6)$$

This expression holds provided that $C(z)$ varies on a scale which is large compared to the inner radius r_i .

Differentiation of equation (2.6) with respect to z gives the second order ordinary differential equation

$$\frac{d^2C}{dz^2} = \gamma^2 C \quad (2.7a)$$

where the constant γ depends on the permeate and tube properties and is given by

$$\gamma^2 = \frac{16\mu}{\alpha r_1^4}. \quad (2.7b)$$

The constant γ has dimensions $[\text{ML}^{-1}\text{T}^{-1}/\text{ML}^{-3}\text{T}^{-1}\text{L}^4]^{\frac{1}{2}} = \text{L}^{-1}$. The general solution of equation (2.7a) is

$$C(z) = c_1 e^{\gamma z} + c_2 e^{-\gamma z} \quad (2.8)$$

and the corresponding expression for the pressure in the lumen is (from 2.2a and 2.5b)

$$p = p_0 - \alpha \{c_1 e^{\gamma z} + c_2 e^{-\gamma z}\}. \quad (2.9)$$

The constants c_1 and c_2 in equation (2.9) are now determined for two cases.

Tube of total length $2X$ ($-X < z < X$)

If the origin $z = 0$ is chosen as the midpoint of the tube, the pressure will be even in z so that $c_1 = c_2$ in (2.9). Also, the pressure at the end of the lumen $z = X$ is p_e which gives

$$p_e = p_0 - 2\alpha c_1 \cosh \gamma X$$

so that

$$c_1 = \frac{(p_0 - p_e)}{2\alpha \cosh \gamma X}. \quad (2.10)$$

Thus the solution for $-X < z < X$ is

$$\begin{aligned} \text{pressure:} \quad p(z) &= p_0 - \frac{(p_0 - p_e) \cosh \gamma z}{\cosh \gamma X}, \\ \text{velocity:} \quad u(r,z) &= \frac{\gamma(p_0 - p_e) \sinh \gamma z}{4\mu \cosh \gamma X} (r_1^2 - r^2). \end{aligned}$$

Also, if the volume flux Q is taken to be zero at the midpoint of the tube, we have that

$$Q(z) = \frac{\pi r_i^4 \gamma (p_0 - p_e) \sinh \gamma z}{8\mu \cosh \gamma X}$$

and the volume flux out of either end of the tube is

$$Q(X) = \frac{\pi r_i^4 \gamma (p_0 - p_e)}{8\mu} \tanh \gamma X. \quad (2.11)$$

Note that the dimensions of the expression are $[L^4 \cdot L^{-1} \cdot ML^{-1}T^{-2} / ML^{-1}T^{-1}] = [L^3T^{-1}]$ (as expected) and that $Q(X)$ tends to the finite value $\pi r_i^4 \gamma (p_0 - p_e) / 8\mu$ as X becomes large. In this limit, the pressure-driven flow through the lumen is limited by the viscous stress at the walls $r = r_i$.

Semi-infinite tube ($-\infty < z < X$)

In this (unrealistic) case, the constant c_2 in (2.9) must be taken as zero and the constant c_1 is determined from the equation

$$p_e = p_0 - \alpha c_1 e^{\gamma X}$$

which holds at the end of the lumen. The solution for $-\infty < z < X$ is

$$\text{pressure:} \quad p(z) = p_0 - (p_0 - p_e) e^{-\gamma(X-z)},$$

$$\text{velocity:} \quad u(r,z) = \frac{\gamma(p_0 - p_e)}{4\mu} e^{-\gamma(X-z)} (r_i^2 - r^2),$$

$$\text{volume flux:} \quad Q(z) = \frac{\pi r_i^4 \gamma (p_0 - p_e)}{8\mu} e^{-\gamma(X-z)}.$$

(Here the volume flux was taken to be zero as $z \rightarrow -\infty$.) Finally, the volume flux at the end $z = X$ of the tube is

$$Q(X) = \frac{\pi r_i^4 \gamma (p_0 - p_e)}{8\mu}$$

which is the limiting value of expression (2.11) for X large.

3. OPTIMIZATION OF THE TOTAL FLUX

Suppose that the filter cartridge contains N hollow fibres. The total flux of permeate from the cartridge is

$$Q_{tot} = 2N Q(X)$$

where $Q(X)$ is given by equation (2.11),

$$Q(X) = \frac{\pi r_i^4 \gamma \Delta p}{8\mu} \tanh \gamma X .$$

The parameters on which Q_{tot} depends are N , r_i , r_o , A , X , μ , k and $\Delta p = p_1 - p_0$. (These parameters are defined in point 1 of Section 1.) Some of the parameters must be regarded as constants (namely μ for a given permeate, k for a given foam, and (A, X) for a given cartridge). There is also the engineering constraint that not too great a fraction of the cartridge cross-section should be occupied by fibres, i.e.

$$\frac{N\pi r_o^2}{A} < \beta_1$$

where $\beta_1 < 1$ is a packing fraction and cannot be so large that the dirty material cannot circulate around the fibres. Further, if β_1 is too large, dirt particles build up between the fibres thereby causing the fibres to become glued together. Under this condition, there is a risk that the fluid pressure acting on the fibres would be sufficient to stretch the fibres beyond their elastic limit. The constant β_1 has to be determined experimentally.

There is another engineering constraint - the pressure drop Δp must not be so large that the fibres would collapse either by buckling or by stresses exceeding a safe working stress such as the elastic limit of the foam. If the critical pressure drop is written $(\Delta p)_{crit}$, the engineering constraint is

$$\Delta p < \beta_2 (\Delta p)_{crit}$$

where $\beta_2 < 1$ has to be assigned by Memtec. Note that $(\Delta p)_{crit}$ depends on r_i and r_o and is examined in some detail in the next two sections.

On combining the above arguments, it can be seen that the tube radii r_i

and r_o need to be chosen to maximize the expression

$$Q_{\text{tot}} = 2 \frac{\beta_1 A \pi r_i^4 \gamma \beta_2 (\Delta p)_{\text{crit}}}{\pi r_o^2 8\mu} \tanh \gamma X \quad (3.1)$$

where, for uniform foams, γ is (equations 2.5b, 2.7b)

$$\gamma = \frac{4}{r_i^2} \left(\frac{k}{\ln[r_o/r_i]} \right)^{\frac{1}{2}}. \quad (3.2)$$

Optimization with respect to r_i , r_o

It is convenient to use the variables

$$\eta = \frac{r_i}{r_o},$$

$$\zeta = \gamma X = \frac{4X}{r_i^2} \left(\frac{k}{\ln[r_o/r_i]} \right)^{\frac{1}{2}}$$

instead of r_i and r_o . Note that $0 < r_i < r_o < \infty$ corresponds to $0 < \eta < 1$, $0 < \zeta < \infty$ and that (r_i, r_o) are given in terms of (η, ζ) by

$$r_i = 2 \left(\frac{X}{\zeta} \right)^{\frac{1}{2}} \left(\frac{k}{\ln[1/\eta]} \right)^{\frac{1}{4}},$$

$$r_o = \frac{r_i}{\eta}.$$

The critical pressure drop $(\Delta p)_{\text{crit}}$ required in equation (3.1) is given by two distinct analyses subsequently in this report, culminating in expressions (4.7) and (5.6). If these equations are re-written using the variables (η, ζ) , the flux is given by

$$\frac{\mu Q_{\text{tot}}}{\beta_1 \beta_2 A k^{\frac{1}{2}}} = \min \left\{ \frac{E}{4(1-\nu)^2} \frac{(1-\eta)^3}{\eta (\ln 1/\eta)^{\frac{1}{2}}}, \right. \\ \left. 2(1-\nu) \Sigma_{\text{max}} \frac{(\ln 1/\eta)^{\frac{1}{2}} \eta^2}{\frac{2 \ln 1/\eta}{1-\eta^2} + 1-2\nu} \right\} \tanh \zeta. \quad (3.3)$$

It is clear from this expression that Q_{tot} can be optimized by treating $\eta = r_i/r_o$ and $\zeta = \gamma X$ separately. To exemplify the calculation, we adopt the values $E = 200$ MPa, $1-\nu^2 = 0.8$ and $\Sigma_{max} = 12$ MPa (see Section 4 for a discussion of these values) and plot on figure 3 the two components in the curly brackets in expression (3.3). For these values of the parameters, it is clear that

$$\max \left\{ \frac{\mu Q_{tot}}{\beta_1 \beta_2 A k^{\frac{1}{2}}} \right\}$$

occurs for η slightly greater than 0.7 and takes the value $2.8 \tanh \zeta$. The maximum value for $\tanh \zeta$ is achieved by taking ζ as large as possible although, in practice, there is nothing to be gained by taking ζ larger than 2. This completes the optimization of Q_{tot} with respect to (η, ζ) and hence with respect to (r_i, r_o) .

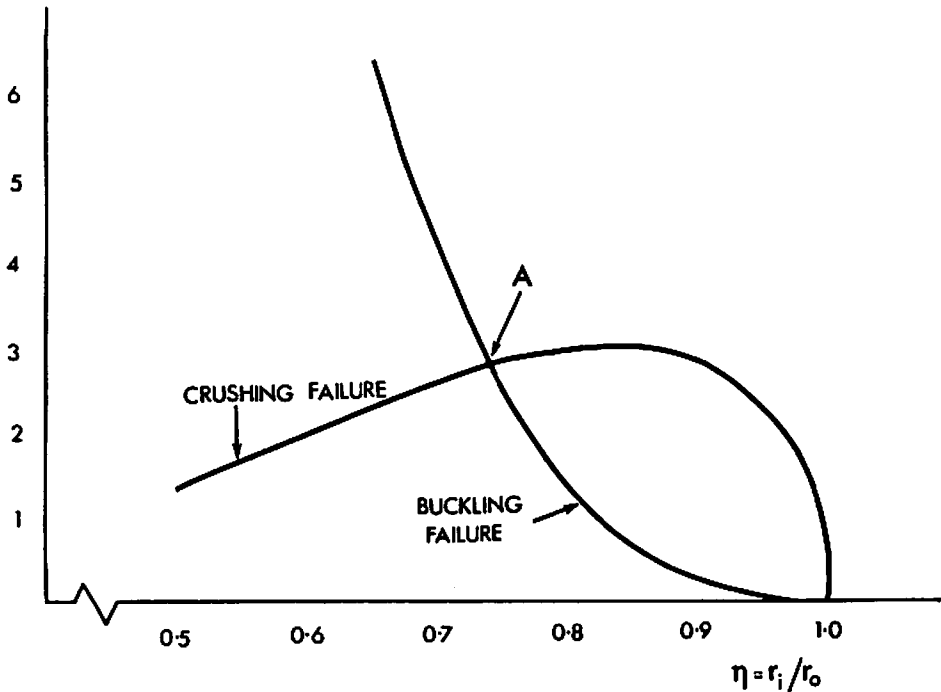


Figure 3: A plot of the η -dependent components of equation (3.3) against η . The point labelled A gives the $\max\{\min\{\text{RHS}\}\}$ with respect to η .

The approach described above depends on the fact that the expressions for $(\Delta p)_{\text{crit}}$ involve (r_i, r_o) only as their ratio r_i/r_o . This must be so (on dimensional grounds) and hence is true for any critical pressure evaluation by a buckling or crushing calculation. A further benefit is that the method allows empirical determination or checking of the optimum ratio of radii for maximum pressure capacity without having to vary two parameters. This could be important in view of the uncertainties in buckling behaviour, especially if the fibres are not exactly circular and concentric.

4. NON-WETTING FAILURE OF THE HOLLOW FIBRES

It was pointed out in the previous section that it is essential to know $(\Delta p)_{\text{crit}}$ - the critical pressure drop across the walls of the hollow fibres at which the fibres will fail. The failure of the fibres can be due to several causes including buckling collapse, the stresses exceeding the yield stress of the material, or the contraction to (nearly) zero of the inner radius under the external pressure. Recall also that a distinction must be made between non-wetting failure (discussed in this section) in which the walls of the fibres do not support a radial fluid flow, and wetting failure in which they do.

Buckling is now examined. A thin-walled, infinitely long cylinder of internal diameter d , thickness t , Young's modulus E , Poisson's ratio ν , is known to collapse by buckling when the pressure drop across the wall exceeds the critical value

$$(\Delta p)_{\text{crit}} = \frac{2E}{(1-\nu^2)} \left(\frac{t}{d}\right)^3$$

(see e.g. Baumeister and Marks, 1958, pp.5-64, 65). In terms of the notation used in this report, this condition becomes

$$(\Delta p)_{\text{crit}} = \frac{E}{4(1-\nu^2)} \left(\frac{r_o - r_i}{r_i}\right)^3. \quad (4.1)$$

This expression is valid provided $(r_o - r_i/r_i)$ is suitably small, say less than 1/5. The expression is the lowest eigenvalue in the problem for the circumferential displacement, and the analysis for this has been simplified by assuming that the radial stress is negligible and the hoop stress varies linearly with radius. These assumptions are inapplicable when the cylinder does not have thin walls and, in this case, the eigenvalue problem for the circumferential displacement

probably has to be solved numerically. Delegates at the MISG were unsure if this case had been analysed, possibly a literature search would reveal the solution. It is noteworthy that buckling is impossible as $(r_i/r_o) \rightarrow 0$, hence

$$(\Delta p)_{\text{crit}} \rightarrow \infty \quad \text{as} \quad r_i/r_o \rightarrow 0. \quad (4.2)$$

There is, however, a well-known solution due to Lamé (1852) for the stress distribution in a thick-walled cylinder in which buckling has not taken place and the radial and hoop stresses are functions of radius only. The solution is (Timoshenko and Goodier, 1951, p.59)

$$\sigma_r = \left\{ \frac{r_i^2 r_o^2 (p_o - p_i)}{r^2} + p_i r_i^2 - p_o r_o^2 \right\} / (r_o - r_i)^2, \quad (4.3a)$$

$$\sigma_\theta = \left\{ -\frac{r_i^2 r_o^2 (p_o - p_i)}{r^2} + p_i r_i^2 - p_o r_o^2 \right\} / (r_o - r_i)^2, \quad (4.3b)$$

in which the boundary conditions

$$(\sigma_r)_{r=r_i} = -p_i, \quad (\sigma_r)_{r=r_o} = -p_o$$

have been applied. The corresponding values for σ_θ at the inner and outer boundaries are found to be

$$(\sigma_\theta)_{r=r_i} = -p_o - 2 \Delta p \frac{r_i^2}{r_o^2 - r_i^2} - \Delta p \quad (4.4a)$$

$$= -p_i - 2 \Delta p \frac{r_o^2}{r_o^2 - r_i^2} \quad (4.4b)$$

and

$$(\sigma_\theta)_{r=r_o} = -p_o - 2 \Delta p \frac{r_i^2}{r_o^2 - r_i^2} \quad (4.4c)$$

in which the notation $\Delta p = p_o - p_i$ has been used. The radial and hoop stresses are sketched in figure 4.

It is clear from figure 4 that the largest stress component is the hoop stress at the inner wall, and, moreover, the thick-walled cylinder may be

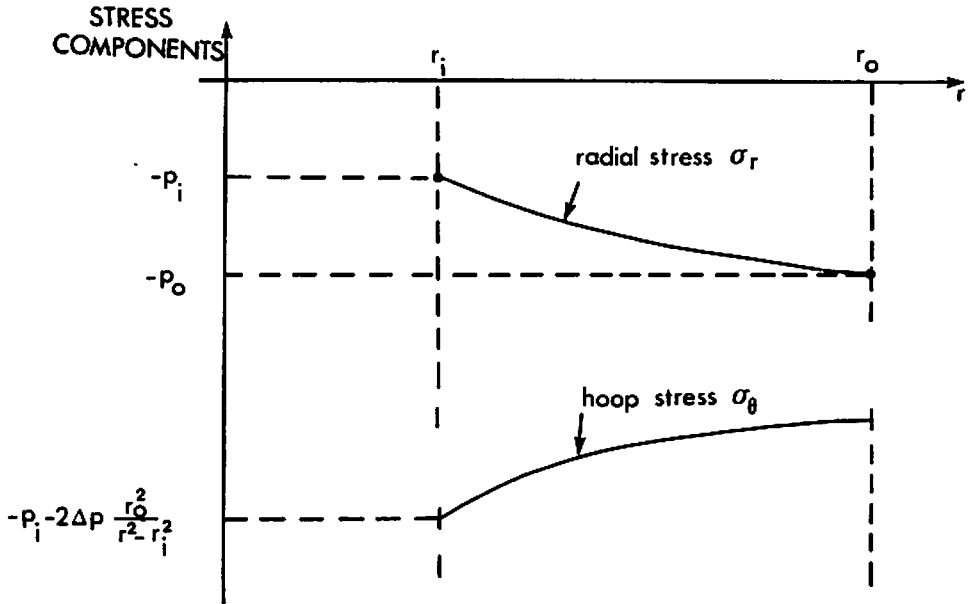


Figure 4: The Lamé solution for the stress components σ_r and σ_θ .

considered to have failed when this component exceeds a safe working value. If the internal pressure p_i in (4.4b) is taken to be zero, failure will occur when

$$2 \Delta p \frac{r_o^2}{r_o^2 - r_i^2} > \Sigma_{\max} \quad (4.5)$$

where Σ_{\max} is the safe working stress of the foam walls.

It is instructive to interpret criteria (4.1, 4.5) diagrammatically. To do this, however, it is necessary to have estimates of the values of the parameters E , ν and Σ_{\max} . These values are not known with certainty. As an example, values of these parameters for a *solid* plastic are assumed as

$$E = 700\text{-}1600 \text{ MPa}, \quad \nu = \text{unknown}, \quad \Sigma_{\max} = 59\text{-}69 \text{ MPa},$$

and the values which were taken for the plastic *foam* were

$$E = 200 \text{ MPa}, \quad 1 - \nu^2 = 0.8, \quad \Sigma_{\max} = 12 \text{ MPa}.$$

Using these values, equation (4.5) yields

$$\frac{4(1-\nu^2)(\Delta p)_{crit}}{E} = \frac{2(1-\nu^2)\Sigma_{max}}{E} \frac{r_o^2 - r_i^2}{r_o^2},$$

that is

$$\frac{4(1-\nu^2)(\Delta p)_{crit}}{E} = 0.096 \frac{r_o^2 - r_i^2}{r_o^2}. \tag{4.6}$$

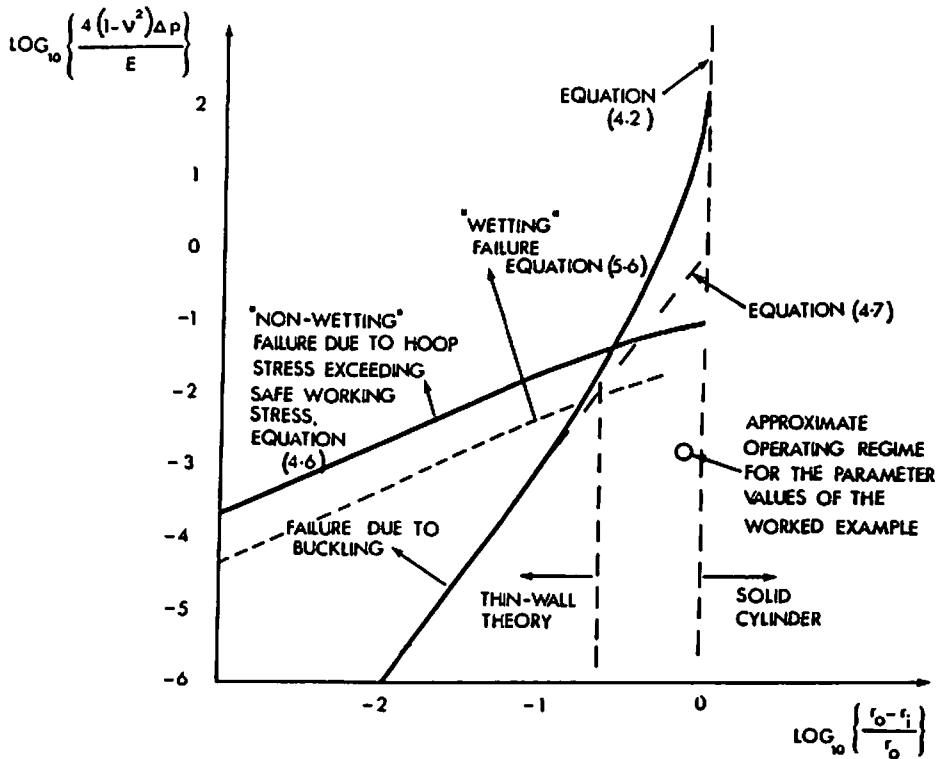


Figure 5: A composite figure to present the essential results of Sections 4 and 5 on the likely critical pressure drop Δp which can be supported by the hollow fibres. In this example, the parameter values are $E = 200$ MPa, $1 - \nu^2 = 0.8$, $\Sigma_{max} = 12$ MPa.

Figure 5 displays all of the above information. Note that equation (4.1) for the buckling of the thin-walled tubes is equivalent to

$$(\Delta p)_{\text{crit}} = \frac{E}{4(1-\nu^2)} \left(\frac{r_o - r_i}{r_o} \right)^3 \quad (4.7)$$

for $(r_o - r_i)/r_i$ small, and this modified form is the one plotted in figure 5. The curve which indicates failure due to buckling has been sketched in a plausible way to connect the two asymptotes (4.2, 4.7). The small area marked on figure 5 corresponds to

$$E = 200 \text{ MPa}, 1 - \nu^2 = 0.8, \Delta p = 100 \text{ kPa}, r_i = 100 \mu, r_o = 300 \mu,$$

and it may be seen that the pressure Δp could be increased by a factor of about 50 before "non-wetting" failure of the tube walls would occur.

5. WETTING FAILURE OF THE HOLLOW FIBRES

We present in this section an alternative model for the stresses in the foam walls. Assume that Darcy's law gives the speed and pressure of the permeate flowing through the foam tube walls. The pressure force which drives the fluid is assumed to have an equal and opposite reaction on the foam material in the walls. The goal is to find this reaction force (or body force) on the foam, and then solve the equilibrium stress and compatibility equations to find the radial and hoop stresses (σ_r and σ_θ respectively).

Equilibrium of radial forces requires that

$$\frac{1}{r} \frac{\partial}{\partial r} (r\sigma_r) - \frac{1}{r} \sigma_\theta + F = 0 \quad (5.1)$$

where F is the body force caused by the permeate flow. A consideration of equation (2.2a) shows that the body force exerted by the permeate is

$$F = -\frac{dp}{dr} = -\frac{\mu C(z)}{rk} \quad (5.2)$$

where $C(z)$ is a slowly varying function of z and, for the largest body forces, is assigned the value at the end of the fibres (see equations (2.5b, 2.8, 2.10))

$$C(z) = \frac{(p_o - p_e)}{\alpha}.$$

Hence, if the permeability k is constant, the body forces near the ends of the fibres are

$$F = - \frac{P_o - P_e}{r \ln(r_o/r_i)}. \quad (5.3)$$

Suppose now that there is zero axial strain (see e.g. Mase, 1970, pp.145-147 for a justification) and let ξ be the radial displacement under the body force (5.3). Linear elasticity gives that the stress components are (see e.g. Timoshenko and Goodier, 1951)

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \left\{ (1-\nu) \frac{d\xi}{dr} + \nu \frac{\xi}{r} \right\}$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)} \left\{ \nu \frac{d\xi}{dr} + (1-\nu) \frac{\xi}{r} \right\}.$$

Equations (5.1, 5.3) then give the equation

$$\frac{d}{dr} \left(r \frac{d\xi}{dr} \right) - \frac{\xi}{r} = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \frac{P_o - P_e}{\ln(r_o/r_i)}$$

for the displacement ξ , and ξ therefore takes the form

$$\xi = \frac{A}{r} + Br + \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \frac{P_o - P_e}{2 \ln(r_o/r_i)} r \ln\left(\frac{r}{r_i}\right), \quad (5.4)$$

where A and B are constants. The stress components are found to be

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \left\{ -(1-2\nu) \frac{A}{r^2} + B \right\} + \frac{P_o - P_e}{2(1-\nu) \ln(r_o/r_i)} \left\{ \ln \frac{r}{r_i} + 1 - \nu \right\}$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)} \left\{ (1-2\nu) \frac{A}{r^2} + B \right\} + \frac{P_o - P_e}{2(1-\nu) \ln(r_o/r_i)} \left\{ \ln \frac{r}{r_i} + \nu \right\}.$$

Now suppose that the pressure acts on the liquid which then exerts a body force on the foam. The appropriate boundary conditions on the radial stress are $\sigma_r = 0$ at $r = r_i$, $r = r_o$. The constants A and B above are therefore found to be

$$A = - \frac{1-\nu}{1-\nu} \frac{P_o - P_e}{2E} \frac{r_i^2 r_o^2}{r_o^2 - r_i^2},$$

$$B = - \frac{(1+\nu)(1-2\nu)}{1-\nu} \frac{P_o - P_e}{2E} \left\{ \frac{r_o^2}{r_o^2 - r_i^2} + \frac{1-\nu}{\ln r_o/r_i} \right\}$$

and these lead to the following expression for the stress components

$$\sigma_r = - \frac{P_o - P_e}{2(1-\nu)} \left\{ \frac{r_o^2}{r_o^2 - r_i^2} \left(1 - \frac{r_i^2}{r^2} \right) - \frac{\ln(r/r_i)}{\ln(r_o/r_i)} \right\}, \quad (5.5a)$$

$$\sigma_\theta = - \frac{P_o - P_e}{2(1-\nu)} \left\{ \frac{r_o^2}{r_o^2 - r_i^2} \left(1 + \frac{r_i^2}{r^2} \right) + \frac{[-\ln(r/r_i) + 1-2\nu]}{\ln(r_o/r_i)} \right\}. \quad (5.5b)$$

It is readily checked that σ_r vanishes at (r_i, r_o) and that the maximum stress is

$$\sigma_\theta|_{r=r_i} = - \frac{P_o - P_e}{1-\nu} \left\{ \frac{r_o^2}{r_o^2 - r_i^2} + \frac{1-2\nu}{2 \ln(r_o/r_i)} \right\}.$$

Thus the condition

$$(P_o - P_e) < (1-\nu) \Sigma_{\max} \left\{ \frac{r_o^2}{r_o^2 - r_i^2} + \frac{1-2\nu}{2 \ln(r_o/r_i)} \right\}^{-1} \quad (5.6)$$

may be derived by ensuring that all stresses are below the safe working stress Σ_{\max} .

Equation (5.6) is plotted in fig. 5 assuming the values $E = 200$ MPa, $1 - \nu^2 = 0.8$, $\Sigma_{\max} = 12$ MPa. The model yields a rather lower critical pressure for failure than does equation (4.6) although it also implies that, for the parameter values which were chosen, the pressure difference $(p_o - p_e)$ could still be increased significantly (by a factor of 10) without causing failure of the hollow fibres.

6. WHEN SHOULD THE HOLLOW FIBRES BE CLEANED?

It is known that the total flux of permeate from the filter cartridge reduces with time as the surface of the hollow fibres becomes clogged with filtrate. According to Memtec, there is strong theoretical and experimental evidence that the flow speed $v(t)$ is given by an expression of the form

$$v(t) = K + \frac{v_0 - K}{1 + Dt^d} \quad (6.1)$$

where v_0 is the initial flow rate, K is the flow rate at very large times, and (D,d) are constants which need to be determined by fitting expression (6.1) to data. It is also known that the constants D,d drift slowly over time, so that expression (6.1) has to be refitted at infrequent intervals.

Now, suppose that it takes a time T_c to clean the filter. It is required to find that time T over which the filter should be operated in order to maximize the average production rate

$$\text{APR} = \frac{1}{T + T_c} \int_0^T v(t) dt. \quad (6.2)$$

At present, this time T_c is determined by a procedure which requires an integration of (6.1) *and* a knowledge of the constants (D,d) . Memtec asked the MISG if a control strategy could be suggested which would avoid these two requirements.

The strategy which was suggested is as follows. Choose the time T so that $d(\text{APR})/dT = 0$, i.e.

$$-\frac{1}{(T + T_c)^2} \int_0^T v(t) dT + \frac{v(T)}{T + T_c} = 0$$

or

$$\int_0^T v(t) dt = (T + T_c) v(T). \quad (6.3)$$

The filter should be cleaned if this condition is satisfied. Note, however, the condition may be written

$$\int_0^T \{v(t) - v(T)\} dt = T_c v(T)$$

and it follows that the condition will never be satisfied if T_c is sufficiently large. (That is, the filter should not be cleaned if T_c is very large.) The condition (6.3) is distribution-free and non-parametric, and it should be an improvement on the present procedure in which volume productivity alone

determines the times of backwash. It still requires, however, an integration with respect to time of the flow rate.

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