Optimal Train Schedule/Train Length

1. INTRODUCTION

This problem concerns a transportation system consisting of a supply of ore (the loading terminal at the Mt Whaleback Mine), a destination (the unloading terminal at Port Hedland), a railway line connecting these locations, rolling stock, crew and other personnel, and a train-and-crew scheduling algorithm. The line traverses more than 400 km and consists of a single track except for a number of train-length sidings and a larger number of short spurs for the use of track maintenance vehicles. Operating costs and capital repayments per unit time must both be considered, and the objective is to determine a policy, complete with detailed schedule, which minimizes total (discounted) cash outflow per unit time.

A typical possibility involves 6 trains per day journeying (*mainly* downhill) from source to destination, each train consisting of 3 locomotives followed by 180 ore cars, each car bearing 105 tonnes of ore. With 60 cars per locomotive, the fully laden train requires assistance (banding) for the relatively short uphill section of the journey (at the Chichester Ranges) - it is pushed by two extra locomotives stationed at the beginning of this section.

Such a train must spend at least its loading time (2 hours) at the source and at least its unloading and routine checking time (5 hours) at the destination. A one-way journey with delays takes 8 hours. Train speed, laden or unladen, is not to exceed 75 km/h and 6 unladen trains must make the return journey each day. To avoid the large costs involved in stoppping and re-starting fully laden trains, the unladen ones must reach sidings at the right times to let the laden ones through. Crews are not attached to particular trains. A crew rest period is to be between 8 and $12\frac{1}{2}$ hours. For it to be possible to schedule the above, we need at least 7 such trains in the system. The one idle train at any time is deemed to require additional capital cost.

Variations other than merely rearranging the daily schedule are possible. For example,we may change the payload (the weight of ore per car); or other train configurations may be used such as 3 locomotives followed by 144 ore cars, or the locotrol configuration with 3 locos at the head of the train and two more in the middle (radio controlled by the single crew at the head of the train, with the aid of a special locotrol unit), with a total of 250 cars - neither of these configurations requires banking. A more drastic variation would be to decide that the scheduling time unit should be other than 24 hours.

The problem turned out to be substantially one of understanding the system and formulating it in such a way that the variables with greater influence on the total cost per unit time could be identified and the scope of the decisions concerning these variables could be sufficiently limited. The few items of reference material available were not directly helpful for this problem, so the following account is self-contained.

2. VARIABLES AND CONSTRAINTS

$K =$ demand at destination (tonnes) per scheduling time unit;

$M =$ mass of an empty ore car (tonnes) - a constant;

$P =$ payload of ore per ore car (tonnes);

$H_i =$ hauling capacity (maximum number of laden ore cars per locomotive) in a train of type i. For head-end configuration, $H_i = 48$ if banking is not used and $H_i = 60$ if banking is used. For locotrol configuration, H_i (no banking) increases to 50.

$L_i =$ number of locomotives in a train of type i;

$C_i =$ number of ore cars in a train of type i;

$T_i =$ number of runs of trains of type i per scheduling time unit;

$U_i =$ number of runs of trains of type i required in the system for a proposed schedule to be feasible.

We also define

$$T = \Sigma T_i , \quad L_t = \Sigma T_i L_i , \quad C_t = \Sigma T_i C_i ,$$

$$U = \Sigma U_i , \quad L_u = \Sigma U_i L_i , \quad C_u = \Sigma U_i C_i ,$$

$$\bar{C} = C_t/T , \quad \bar{T} = T_t/T .$$

$$\mathbf{T} = (T_1, T_2, \ldots). \quad \mathbf{U} = (U_1, U_2, \ldots) .$$

[If all trains are of the same configuration, the above definitions simplify: e.g. $L_t = TL$.]

The following constraints apply:

(a) $PC_t \geq K$ - we shall assume that P and/or C_t have been adjusted to make $PC_t = K$.

(b) $90 \leq P \leq 125$ - it is not worth having $P < 90$, and ore would spill with $P > 125$.

(c) $C_i \leq H_i L_i$, and $C_i \leq 255$ (for trains to fit onto existing sidings).

(d) If the scheduling time unit is 24 hrs, then $T \leq 10(8)$ if banking is never (always, resp.) used.

Note: If indications are that it is better to keep T small (and the requirements of permanent way maintenance favour this), then the scheduling constraint posed by the limited set of sidings is not severe.

General Warnings

(1) A train for which $C > 144$ is split into two parts for unloading, and this affects the unloading time and thereby the scheduling problem. Cases $C \leq 144$ and $C > 144$ are therefore basically different.

(2) If there are two or more types of train being run, then in cases where an extra train is needed in the system, we may well need an extra train of each type being used.

(3) It has been assumed that the configuration of each train, once the system is running, is left unchanged. It is not clear that this strategy is consistent with optimality.

(4) Given T_i and L_i, we may sometimes reduce U_i by reducing C_i (details later). Thus the standard configurations may need to be more numerous than those commonly considered.

3. COSTS

(A) Capital costs

Purchase prices: \$x per locomotive, \$y per ore car, \$z per locotrol unit and \$w per km of new track (e.g. for an extra siding, if this is to be considered). Here $x > w > z > y$ - in each inequality, the larger term is more than twice as large as the smaller.

If an interest rate of $100\rho\%$ applies, then the annual cash outflow in repayments for a locomotive, for example, must exceed $\$x\rho$. Company policy will fix a repayment rate r (> 1), resulting in an annual cash outflow of $\$x\rho r$ per locomotive, etc. The evaluation of such outflow in relation to inflation, tax, etc., as compared with the evaluation of operating cost figures, will also be determined by the Company.

In any case the quantity $(L_u x + C_u y)\rho/365$ needs to be considered, and to be expressed in terms of the unit C^o_{op}, the current daily operating costs. (Corresponding terms for the proposed number of locotrol units and the porposed length of new track (if any) must also be included.)

(B) Operating costs

Currently, 70% of the total operating cost comes from $\alpha,\ldots,\epsilon$ below - α making the greatest single contribution, β and γ each making a substantial contribution, but δ and ϵ being relatively small. Our attempts at minimizing total daily costs will therefore, in the first instance, focus on those variables which most affect α, β and γ.

α: permanent way maintenance cost:

α_1 = contribution via T (as T increases, the track becomes less accessible to maintenance vehicles);

α_2 = contribution via C_t, M and M+P (wearing out of track).

β: locomotive maintnenace cost - assumed to be a constant multiple of L_t.

γ: locomotove fuel cost - assumed to depend on C_t, M and M+P.

δ: crewing cost - a function of T (which can take into account extra costs of banking and for splitting longer trains for unloading).

ϵ: ore-car maintenance cost - a function of C_t, M and M+P.

Let S be the scheduling time unit (cycle length), measured in days. Then C_{op} is defined as $(\alpha + \beta + \gamma + \delta + \epsilon)/S$. Currently $S = 1$ and C_{op} is 70% of the total daily operating costs. C_{op} necessarily represents an amount of money per day.

Our task is to provide a procedure for minimizing, or at least reducing, $C_{cap} + C_{op}$.

Notes:

(1) Relative contributions of "up" (unladen) and "down" (laden) journeys need to be determined for α_2, γ and ϵ (in decreasing order of importance): apart from the proportionately larger amount of fuel used in the mainly uphill journey, there is also the effect of the rattling of the empty ore cars. We expect that the relative contribution of the "up" journey will be greater in the case of γ than with α_2 and ϵ.

(2) In α_2 and γ the relatively small contributions of the locomotives have been neglected. This means that the estimates of cost changes below (as P is varied) will be systematically a little too large.

(3) α_2, γ and ϵ are each of the form C_t (function of M + function of (M+P)) where $C_t = K/P$. At the lower levels of the optimization procedure, S and K are taken as constants, so, for any given value of P, we may try to optimize $\alpha_1 + \beta + \delta +$ appropriate capital costs. But see later (Cost Minimization, Capital Costs), where results show that a decrease in T, leading to an increase in $\bar{C}$, leads to an increased average turnaround time for trains, which may increase U and the capital costs. Further, the question of smoothing the work load at source and destination (if this is desirable) would lead us to consider larger T rather than smaller.

(4) If L_u and C_u are increased while L_t and C_t remain constant, β and ϵ *may* increase slightly.

(5) The total mass hauled on down-journeys in each time interval S is $K + MC_t = K + MK/P$, a decreasing function of P.

4. COST MINIMIZATION

Optimization techniques for discrete and for continuous variables are quite different, particularly where the discrete variables can assume only a small number of values. Thus, for example, the question of how many trains (if any) to bank is treated, not by a formula, but by the enumerator of separate cases in the final search technique.

The operating and capital cost analyses also differ markedly. Thus, with $\bar{C} = C_t/T$, and taking K as a constant, the constraint $P\bar{C}T$ ($= PC_t$) = K allows the elimination of one variable, but C_{op} is more readily handled in terms of P and is related to C_t, while C_{cap} is more readily handled in terms of the C's and is related to L_u and C_u, and finally, **U** is determined from **T** by the requirements of train and crew scheduling.

Dimensionless formulation of operating costs

Let $f_{11}, f_{12}, f_2, \ldots, f_5$ be the fractions of the total daily operating cost currently contributed by $\alpha_1, \alpha_2, \beta, \ldots, \epsilon$ respectively, and let $f_1 = f_{11} + f_{12}$.

It is known that $f_1 > f_2 \simeq f_3 > f_4 \simeq f_5$ and that $f_{12} > f_{11}$, each inequality being substantial.

Let $K^o, P^o, \ldots$ denote current values, and let $k = K/K^o$, $p = P/P^o$, ... for any proposed plan. Take as unit of money the present daily operating cost. Then the following results are derived:

$$\alpha_2 = f_{12}\left\{(1-\eta)c_t + \eta c_t\, f\left(\frac{M+P}{4}\right) \Big/ f\left(\frac{M+P^o}{4}\right)\right\} \tag{i}$$

where $\frac{M+P}{4}$ is the axle load of a laden ore-car, and η is the fraction of α_2 contributed by the laden trains on their "down" journey.

It is suggested that $f(x) = x^n$ for some $2 \leq n \leq 3$. Note that $c_t = C_t/C_t^o = K/P \div K^o/P^o = k/p$. Let $\theta = M/P^o = 25/105 \simeq 1/4$. Then

$$\frac{M+P}{M+P^o} = \frac{\theta+p}{\theta+1},$$

so that we suggest

$$\alpha_2 = f_{12} \cdot \frac{k}{p} \cdot \left\{ 1\text{-}\eta+\eta \left(\frac{\theta+p}{\theta+1}\right)^n \right\}$$

so that

$$\left.\frac{\partial \alpha_2}{\partial p}\right|_{p=1} = f_{12}k \left\{ \frac{\eta n}{\theta+1} - 1 \right\} . \qquad \text{(i)}'$$

Estimates:

For example $\eta \simeq 0.85$, $n \simeq 2$, $\theta \simeq \frac{1}{4}$ gives

$$\frac{\eta n}{\theta+1} - 1 = \frac{1.7}{1.25} - 1 = 1.36 - 1 ,$$

a relatively small difference of larger quantities.

Because of the importance of α_2, we need to know η and n fairly accurately, and care is necessary if $|p\text{-}1|$ is too large, since then $\alpha_2 - \alpha_2^o$ may not be well approximated by

$$(p\text{-}1) \left.\frac{\partial \alpha_2}{\partial p}\right|_{p=1} .$$

Note:

If the rattling of empty cars increases their contribution to wear on the rails by a factor r, and if the formula $f(x) = x^n$ applies even when the axle load is just M/4, then

$$1\text{-}\eta = \frac{r\theta^n}{r\theta^n + (1+\theta)^n} = \frac{r}{r + (1 + 1/\theta)^n} .$$

With $n = 2$ and $\eta = 0.85$, we get r to be almost 5, which is surprising.

The term α_1 is less well known. One guess is that, for $4 \leq T \leq 10$, $\alpha_1(T)$ can be approximated by a quadratic function of T with a stationary value at $T = 4$, so that then

$$\alpha_1 = f_{11} \cdot \frac{(T\text{-}4)^2 + \lambda}{(T^o - 4)^2 + \lambda} , \text{ for some constant } \lambda > 0 .$$

$$\gamma = f_3 \cdot \frac{k}{p} \cdot \left\{1 - \eta' + \eta' \cdot \frac{\theta+p}{\theta+1}\right\} \qquad \text{(ii)}$$

where η' is the fraction of γ contributed by the laden trains on their "down" journey.

Then

$$\left.\frac{\partial\gamma}{\partial p}\right|_{p=1} = f_3 k \left\{\frac{\eta'}{\theta+1} - 1\right\} \qquad \text{(ii)}'$$

< 0 since $\eta' < 1$.

The term ϵ is treated similarly:

$$\epsilon = f_5 \cdot \frac{k}{p} \cdot \left\{1 - \eta'' + \eta'' \cdot \frac{\theta+p}{\theta+1}\right\} \text{ for some constant } m \geq 1 . \qquad \text{(iii)}$$

Then

$$\left.\frac{\partial\epsilon}{\partial p}\right|_{p=1} = f_5 k \left\{\frac{\eta'' m}{\theta+1} - 1\right\} . \qquad \text{(iii)}'$$

so that it is important to know m and η'', even though f_5 is smallish.

Overall, $\frac{\partial}{\partial p}(\alpha_2 + \gamma + \epsilon)$ can be estimated as a function of p and, in particular, at $p = 1$.

Present indications are that $\alpha_2 + \gamma + \epsilon$ is increasing, as a function of p, but more precise numerical data will probably be needed. (We expect that η' will be smaller than either η or η''.) Tentatively, $\alpha_2 + \gamma + \epsilon$ will be decreased if we decrease p.

Notes:

(1) In a term such as ϵ, if, as a first approximation, we omit consideration of the "up" journey (unladen) and the presence of M, then we obtain a non-decreasing function of p, with

$$\left.\frac{\partial \epsilon}{\partial p}\right|_{p=1} = f_5 k(m-1).$$

However, if the "up" journey and M are taken into account, we find that $\left.\frac{\partial \epsilon}{\partial p}\right|_{p=1}$ may be < 0 even if m > 1.

(2) Other terms (α_1, β ad δ):

$\alpha_1(T)$ is conjectural at this stage, as already noted;

$\beta(L_t)$ is probably best left as a function of L_t which, together with **T** (from which it is determined), places bounds on C_t;

$\delta(\mathbf{T})$ must take into account crews for banking and the possible use of charter flights for transporting crews, together with any extra personnel expenses that may be involved in the unloading of longer trains (which are split into 2 sections for this purpose).

5. CAPITAL COSTS

Here we may easily generate *lower bounds* for rolling stock requirements, but precise requirements depend on detailed scheduling. The lower bounds could assist in the final algorithm, where it would be useful if most unsatisfactory possibilities could be rejected before an attempt was made at scheduling them. While available rolling stock comfortably exceeds requirements, the following considerations of capital cost are of lesser importance - they would assume greater importance if replacement and/or major overhaul were needing a fresh decision, or if the question arose of selling off or re-deploying some surplus stock.

Lower bounds on rolling stock requirements (L_u, C_u and locotrol units) will come from a knowledge of turnaround time A as a function of C and a sample method of (under)estimating requirements given T, $\bar{C}$ and A.

Warning:

If a schedule is too tight, then apart from the effects of any breakdown there is the danger that if K is to be increased *slightly* this will be done simply by increasing P (C_t already being at the upper limit of some range), with consequent increase in α_2 (and ϵ?). For these reasons, together with the fact that the method employed may sometimes underestimate requirements, we are less worried if calculations are based on what may be a slight overestimate of turnaround time.

Definition

The *turnaround time* for a train is the sum of the times for loading, journey to port, unloading and routine servicing, and return to mine.

Each one-way journey occupies 8 hrs, the loading time is $C/72$ hrs, and the unloading and routine servicing time is $3 + C/72$ hrs if $C \leq 144$ and $5\frac{3}{4} + C/144$ hrs if $C > 144$. Trains with $C > 144$ are split into two sections and the two sections are then unloaded simultaneously. Denoting the turnaround time by $A(C)$, we have:

$$\text{for } C \leq 144, \quad A(C) = 19 + C/36 \leq 23, \quad \text{which is} < 1 \text{ day}$$

$$\text{for } C > 144, \quad A(C) = 21\tfrac{3}{4} + C/48 > 24\tfrac{3}{4}, \quad \text{which is} > 1 \text{ day} .$$

(If there are trains of several types involved, we may define mean turnaround time $\overline{A(C)} = \Sigma\, A(C_i)/T$. Then $\overline{A(C)} = A(\bar{C})$ iff all trains have $C \leq 144$ or all trains have $C > 144$, but in mixed cases $A(\bar{c})$ under-(over-)estimates $\overline{A(C)}$ according as $\bar{C} \leq (>)144$. Here we have assumed that C is necessarily $< (21\frac{3}{4} - 19)/(\frac{1}{36} - \frac{1}{48}) = 396$, which is so).

Let $U' = U\text{-}T$ in some proposed scheme. Then comparing $S + (U'/T)S$ with $\bar{A}$ is the same as comparing U with $(\bar{A}/S)T$. If, in either comparison, the LHS exeeds the RHS by a sufficiently comfortable margin, then there is reason to hope that U is sufficient for a feasible schedule to exist. In any case, the smallest integer greater than or equal to $(\bar{A}/S)T$ will be denoted by **u** and it is a lower bound for U.

For example, with $S = 24$ hrs, $K = 10^5$ tonnes and $P = 105$ tonnes, we have $C_t = 953$ cars/day. If it is proposed that there shall be 5 train runs per 24 hours, all trains being of the same configuration, then $C = 953/5 = 191$ (rounding up) so that, without banking, we require $L = 4$. Then $A(C) = 21\frac{3}{4} + \frac{191}{36} = 27\frac{1}{18}$ so that $(A/S)T \simeq 5\frac{5}{8}$, giving $\mathbf{u} = 6$ with a fairly comfortable margin above $(A/S)T$, but detailed scheduling may well involve $U = 7$.

In general, with $S = 24$, we have

$$\text{for } C \le 144, \quad A = 19 + C/36 ,$$

so

$$(A/24)T = \frac{19}{24}T + \frac{C_t}{24 \times 36}$$

$$\text{for } C > 144, \quad A = 21\tfrac{3}{4} + C/48 ,$$

so

$$(A/24)T = \frac{24}{32}T + \frac{C_t}{24 \times 36} .$$

Here we have taken $C = C_t/T$. In fact C should be rounded up to an integer, so that C_t (= TC) should be rounded up to a multiple of T. However, the small fractional error in C (given that C is of order 100) will lead to no significant error in u.

Warning:

In the case of T (or of U), failure to round up to an integer value may lead to errors. For example consider $S = 24$, $K = 10^5$, $P = 105$, $C_t = 953$ and a decision to take $C = 96$ (with $L = 2$) for as many trains as are needed. Here $A = 19 + \frac{96}{36} = 21\frac{2}{3}$. Taking $T = \frac{953}{96}$, we got $(A/S)T = 8.962$, with the hope that we can take $U = 9$. However T must be rounded up first to 10, and then $(A/S)T = 9.028$, so that $U = 9$ is not feasible (unless, of course, we increase P to reduce C_t).

In general, if all trains are to be of the same configuration, then the *total number of ore cars required in the system*, UC, is a *piecewise increasing* function of C, since U is an integer-valued step function of C; at the points of discontinuity, there is a drop of U and thence a drop of UC. If T and U are not constrained to be integers, we obtain (in the relevant range of C) a *continuous decreasing* function of C (except for the discontinuity of $C = 144$ where the unloading procedure changes)! This would change the whole nature of the argument.

Conclusions to Considerations in Cost Minimization

(a) Since the calculation of **u** involves rounding up (at two stages if C is taken as the decision variable) and since rolling stock requirements can be reliably determined only by detailed scheduling, it is probably not fruitful to proceed with analysis of *formulae* that attempt to relate C, T and U.

(b) If $\alpha_2 + \gamma + \epsilon$ is an increasing function of P, then any P under consideration may be reduced so long as no increase in T or U occurs. So, for any proposed **T** and **U**, take **C** to minimize P subject to not changing **T** and **U**.

(c) There should be sufficient lee-way in any scheme for the effects of a single breakdown or other single disruption not to involve a breakdown of the whole schedule.

6. RECOMMENDATIONS AND ALGORITHM

(1) Since crewing considerations and handling at mine and port may make S = 24 hours a constraint, optimization with respect to S will probably not be required.

(2) For any proposed **T** and U, we maximize the number of ore cars used, subject to restrictions of hauling capacity and subject to restrictions on turnaround times because of **U**. This will produce a *small* number of possible train configurations from which to make a selection.

The algorithm is a *nested optimizing search*, major decisions (purchase of an extra locomotive, construction of a new siding, ...) being placed in the "outer" parts of the structure, lesser decisions (e.g. selection of train types to be used, *given* a supply of rolling stock) being placed next "in", and the finer detail (e.g. the numbers of ore cars per train, *given* the numbers of the various train types) being placed at the "centre".

The algorithm below has the advantage that it produces a range of *near-optimal* solutions as well as the optimal one(s). It may also indicate, from "inner" optimization calculations, whether a (relatively) major decision is worth examining. For example, it may be impossible to schedule an otherwise economical scheme because there are not enough locotrol units. In this case the inner part of the algorithm should be re-run with an extra locotrol unit available (and costed).

CONCLUDING REMARKS

The appropriate "solution" to the problem was *not* a list of numbers of train configurations with an explicit schedule for trains and crews: those who presented the problem are able to generate schedules, and the details of the problem are not fixed once and for all (e.g. demand may increase, or a new supply point might be established and a line from it to the existing line would have to be constructed).

The "solution" consisted of formulating the problem in such a way that the Mt Newman Mining Co. could be given a procedure whereby good (optimal?) decisions could be made and revised in changing circumstances.

The mathematics in the recommended procedure was not difficult - the most difficult calculation was finding a partial derivative (*or*, finding a linear approximation to a non-linear function in a neighbourhood of a suggested point). The contribution of the mathematicians involved was to understand the system and, from the available techniques of analysis and optimization, to select appropriate ones and fit them into an overall strategy.

Nested optimization algorithm

Given a combination of major resources (terminals, track)

Given a combination of less major resources (sidings, locomotives)

Given a combination of still less major resources (control units, ore cars (considered individually))