Color Matching

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Abstract

We consider the problem of creating paint of a certain target color by mixing colorants. Although a large number of colorants is available, in practice it is only allowed to use a limited number. We focus on the problem of selecting the right subset of colorants.

Keywords

Linear programming, Paint mixing, Kubelka-Munk model, Greedy algorithm.

1 Introduction

Consider the following problem. A car enters a garage for repair. The paint layer of the car has been damaged. We want to repair the damage without completely repainting the whole car. To remove every trace of damage, we will locally apply paint of a color very similar to the color of the paint that is already on the car. The problem is, where do we get paint of the right color? Usually, such paint will not be available ready-made. We need to create it ourselves by the familiar process of mixing colorants.

In the next section we describe a simple model due to Kubelka and Munk that allows us to predict the color of a mixture of colorants, given a recipe specifying that colorant i is used in relative proportion c_i.

Using this model it is possible, given a target color and a set of colorants, to compute a recipe that produces a best approximation to the target color.

This seems to solve our initial problem completely (assuming that the Kubelka-Munk model is accurate), but there is a catch. For several reasons, we do not want to use too many colorants in a recipe, not more than k say, whereas there are many more colorants available, say n. We need to select, given our target color, a good set of k colorants to use in our recipe. In fact, we are interested in a couple of k-sets that produce a good approximation to the target color.

Of course, we could compute a recipe for each k-set of colorants, and then decide which k-sets produce the best approximations to our target color. Since computing one recipe is already nontrivial, and n over k will be an exceedingly large number, this takes too much time. Moreover, many of the k-sets will only produce very poor approximations to the target color (k shades of blue will never make a good red), and it seems wasteful to precisely compute many recipes when only a few good ones are needed.

In this paper, we will explain an approach that could be used to wield out bad k-sets without computing many recipes.

2 The Kubelka-Munk model

For a given painted surface and a wavelength λ, the *reflectance* $R(\lambda)$ is defined as the proportion of light of wavelength λ that is reflected by the paint layer. The *color* of the surface is determined by the reflectance values of light in the visible spectrum.

Colorants have two parameters, the *absorption* $K(\lambda)$, and the *scattering* $S(\lambda)$, both depending on the wavelength λ. We may assume that we know both parameters for each of our colorants.

The Kabulka-Munk model predicts that a completely hiding paint layer will satisfy the following relation between the reflectance and the parameters of the colorant, for each wavelength λ:

$$\frac{K(\lambda)}{S(\lambda)} = \frac{(1 - R(\lambda))^2}{2R(\lambda)}. \tag{1}$$

Moreover, when we mix colorants $1, \ldots, k$ in relative proportions c_i (so $\sum_i c_i = 1$ and $c_i \geq 0$) we have

$$K(\lambda) = \sum c_i K_i(\lambda), \tag{2}$$

and

$$S(\lambda) = \sum c_i S_i(\lambda), \tag{3}$$

where K_i, S_i are the coefficients of colorant i and K, S are the coefficients of the mixture.

In practice, we consider only a finite number of frequencies $\lambda_1, \ldots, \lambda_l$ adequately representing the visible spectrum. That is, we measure the reflectance values $R_t(\lambda_1), \ldots, R_t(\lambda_l)$ of our target color. A mixture of colorants is considered a very good approximation if it has the same reflectance values at wavelength $\lambda_1, \ldots, \lambda_l$. Let us say that a set of colorants I is *very good* if there is a recipe using only colorants in I that gives a very good approximation of the target color.

From (**??**) – (**??**) we derive that a set I is very good if and only if there exist $c_i \geq 0$ such that:

$$\frac{\sum_{i \in I} c_i K_i(\lambda_j)}{\sum_{i \in I} c_i S_i(\lambda_j)} = \frac{(1 - R_t(\lambda_j))^2}{2R_t(\lambda_j)}, j = 1, \ldots, l. \tag{4}$$

Since the human eye does not perceive color with the precision of a spectrometer, a 'very good' set of colorants is in fact more than we need. But let us concentrate on very good sets of colorants for now.

3 A geometrical view

Rewriting (**??**) we obtain the following. The set I is very good if there exist $c_i \geq 0$ such that

$$\sum_{i \in I} c_i w_i = 0, \tag{5}$$

where the w_i are vectors in $\mathbb{R}^l$ whose j-th coordinate is defined by

$$(w_i)_j := K_i(\lambda_j) - \frac{(1 - R_t(\lambda_j))^2}{2R_t(\lambda_j)} S_i(\lambda_j). \tag{6}$$

Note that the vector w_i is completely determined by the parameters of the colorant i and the reflection values of the target color.

Equation **??** has a simple geometric interpretation: it states that I is a very good set if and only if the origin is in the convex hull of $\{w_i \mid i \in I\}$.

Now remember that our goal is to limit the size of the set of colorants used in the recipe, i.e. limit the cardinality of I by k. So when we look for very good sets of colorants, we are faced with the following geometrical problem:

> Given a set of vectors in $\mathbb{R}^l$, select a subset of at most k vectors whose convex hull contains the origin.

If the vectors $w_1, \ldots, w_n$ are in general position, a subset of these vectors containing the origin in its convex hull will have at least $l + 1$ elements. In other words, there are no very good sets of cardinality $\leq l$. This is a problem, since k is usually less than l in our application, and there is no reason why the vectors shouldn't be in general position.

It is time to use the fact that the human eye can be fooled, and determine when a set of colorants is good enough.

4 The eye

On the retina of the human eye there are light sensors of three types, each type maximally sensitive to light of a distinct wavelength. Light entering the eye will stimulate sensor of type t ($t = 1, 2, 3$) proportional to

$$z_t := \sum_j A(\lambda_j) a_{jt} \tag{7}$$

where $A(\lambda)$ is the absolute intensity of light of wavelength λ_j entering the eye and a_j is the relative sensitivity of type t to light of wavelength λ_j. The vector $z := (z_1, z_2, z_3)$ is all the information the brain gets from the entering light: thus our color sense is essentially 3-dimensional, and there is a linear map $Z : (A(\lambda_j))_j \mapsto z$.

For any fixed z, the eye is unable to distinguish between any two kinds of light with absolute intensity vectors in $Z^{-1}(z)$.

The color of light emitting from a painted surface depends on both the reflection values $R(\lambda)$ of the paint layer and the environmental light illuminating the surface. By definition of R, we have $A_{out}(\lambda_j) = R(\lambda_j) A_{in}(\lambda_j)$ for each wavelength λ_j. So given a certain kind of environmental light e, we have a linear map $Y_e : (R(\lambda_j))_j \mapsto (A_{out}(\lambda_j))_j$. This somewhat enhances the ability of the eye to distinguish paint colors. Two paint layers, with reflection vectors r_1, r_2 can appear to the eye to have the same color in one kind of environmental light ($Z(Y_1(r_1)) = Z(Y_1(r_2))$), but can be seen to have a different color under another kind of light ($Z(Y_2(r_1)) \neq Z(Y_2(r_2))$). This phenomenon is known as *metamerism.*

In practice, a car is not looked at under every possible kind of light, and this makes our job somewhat easier. We may assume that the repaired car will only be scrutinized in a very limited set of environments, say in daylight and in the light that is usually emitted by street lamps. This means that if the paint layer on the car has reflection vector $r_t := (R_t(\lambda_1), \ldots, R_t(\lambda_l))$, it is satisfactory if we create paint with reflection vector r such that

$$Z(Y_{daylight}(r)) = Z(Y_{daylight}(r_t)), \tag{8}$$

and

$$Z(Y_{streetlight}(r)) = Z(Y_{streetlight}(r_t)) \tag{9}$$

This procuces a set $\mathcal{R} \subseteq I\!R^l$ of reflection vectors that can be safely substituted for the target reflection vector r_t. The solution set of (**??**) is an affine subspace of $I\!R^l$ but note that reflection values should be between 0 and 1. We may even want to restrict ourselves to $R(\lambda_j)$ between $R_t(\lambda_j) \pm \epsilon$. In any case, $\mathcal{R}$ will be a convex set.

We will say that set of colorants I is ***good enough*** if by mixing colorants from I we can create paint with reflection vector $r \in \mathcal{R}$.

From the previous section it follows that I is good enough if and only if there is some reflection vector $r = (R(\lambda_1), \ldots, R(\lambda_l)) \in \mathcal{R}$ such that

the origin lies in the convex hull of $\{w_i^r \mid i \in I\}$,

where

$$(w_i^r)_j := K_i(R(\lambda_j)) - \frac{(1 - R(\lambda_j))^2}{2R(\lambda_j)} S_i(\lambda_j). \tag{10}$$

Thus $w_i = w_i^{r_t}$.

5 Two methods

5.1 The random hyperplane method

Consider a finite set of vectors U in $I\!R^l$. It is clear that if U is strictly on one side of a (linear) hyperplane H, then the convex hull of U does not contain the origin. From geometrical intuition it is also obvious (but not trivial to prove) that the converse holds: namely that if the convex hull of U does not contain the origin, there is some hyperplane H having all of U strictly on one side. Such is the content of Farkas' Lemma (see e.g. [?]):

Lemma 1 (Farkas) *Given a finite set of vectors $U \subseteq \mathbb{R}^l$ exactly one of the following statements hold:*

1. *0 lies in the convex hull of U, and*
2. *there is a vector $d \in \mathbb{R}^l$ such that $(d, u) > 0$ for all $u \in U$.*

By $(.,.)$ we denote the inner product of two vectors.

This can be put to use for our problem in the following way.

Let us first consider 'very good' sets of colorants again. If we take an arbitrary vector $d \in \mathbb{R}^l$, and set $F_d := \{i \mid (d, w_i) > 0\}$, then it follows from (the easy part of) Farkas' Lemma that any set of colorants I with $I \subseteq F_d$ will not be a very good set. The nontrivial part of Farkas' Lemma shows that any set that is not 'very good' has a nonzero chance of being a subset of F_d . We may rapidly construct a multitude of such 'forbidden' sets, by randomly choosing vectors $d_1, \ldots, d_N$ from $S^{l-1} := \{d \in \mathbb{R}^l \mid \|d\| = 1\}$. Then we search for k-sets I that satisfy $I \setminus F_{d_i} \neq \emptyset$ for all $i = 1, \ldots, N$, and provided that N is big enough such an I will very likely be a very good set.

If we are interested in sets that are 'good enough', the problem becomes more subtle. Define for each colorant i the set $W_i := \{w_i^r \mid r \in \mathcal{R}\}$ where $\mathcal{R}$ is the set of safe substitutes for the target reflection vector r_t of the previous section. Given any vector $d \in \mathbb{R}^l$, we put

$$F_d := \{i \mid \min_{w \in W_i} (d, w) > 0\}. \tag{11}$$

Clearly, no subset of F_d will be good enough. It is not true anymore that any set that is not good enough is eliminated this way. Still, we can construct many such forbidden sets F_d each time killing many candidate k-sets of colorants.

The minimization problem $\min_{w \in W_i} (d, w)$ is hard in general but

1. we may assume that $\mathcal{R}$ is a polytope, and
2. we can replace w_i^r by its linear approximation around $w_i^{r_t}$ provided that $\|r - r_t\|$ is small,

yielding a polytope $\tilde{W}_i$ approximating W_i. We can either solve the minimization problems $\min_{w \in \tilde{W}_i} (w, d)$ for each d or compute the vertices V_i of $\tilde{W}_i$ in advance, and use the fact that $\min_{w \in \tilde{W}_i} (w, d) = \min_{v \in V_i} (v, d)$ for every d.

A faster, but more crude method is to replace the condition

$$\min_{w \in W_i} (d, w) > 0$$

in (**??**) by $(d, w_i) > \epsilon$ for some strategically chosen $\epsilon > 0$. Thus we use the unquantified notion that we can still displace the vectors w_i, but only a little. If a set of vectors is far on one side of a linear hyperplane, the chances that a small displacement of these vectors has the origin in its convex hull become very thin.

When a suitable collection of forbidden sets $F_1, \ldots, F_N$ has been constructed, it remains to find sets I such that $I \not\subseteq F_i$ for all i. Equivalently, we want an I such that $I \cap \overline{F_i} \neq \emptyset$ for all i, where $\overline{U}$ denotes the complement of a set. This is known in the literature as a *set covering problem* : the set I needs to 'cover' each $\overline{F_i}$.

Although there is no direct relation to the current problem, the approach described in this section was inspired by the method described in [?].

5.2 The greedy algorithm method

The greedy algorithm is a very general method to pick a good k-subset I out of an n-set C. To apply the greedy algorithm we need a measure f of how good a subset is. For the moment we will not specify this function, we only remark that it acts on all subsets of C and its image is a real value. Applied to our problem, this algorithm looks like this:

1. $I \leftarrow \emptyset$; $c \leftarrow 0$.
2. choose $i \in C$ such that $f(I \cup \{i\})$ is maximal
3. $I \leftarrow I \cup \{i\}$; $c \leftarrow c+1$
4. IF $c = k$ THEN RETURN(I)
5. goto 2.

In step 2 we intentionally do not specify that $i \notin I$. In that way I is increased with some element only when this results in an improvement.

The main advantage of this algorithm is that it runs very fast. The speed depends heavily on how fast can we evaluate the function f. The drawback is that we may end with a far from optimal subset I. We will start very well picking the first elements but the subsequent choices made in step 2 can be very weak. One solution to this problem is to incorporate some flexibility (an integer m will be the measure of flexibility). Now we give a second version of the greedy algorithm with flexibility m:

1. $I_1 \leftarrow \emptyset; \cdots I_m \leftarrow \emptyset$; $c \leftarrow 0$
2. find m elements $(i_1, j_1), \ldots, (i_m, j_m)$ in $C \times \{1, \ldots, m\}$ that take the m maximal values (in order) of the function $(i,j) \rightarrow f(I_i \cup \{j\})$ and such that for all $1 \le s, t \le m$, $I_{i_s} \cup \{j_s\} \neq I_{i_t} \cup \{j_t\}$.
3. $I_1 \leftarrow I_{i_1} \cup \{j_1\}, \ldots, I_m \leftarrow I_{i_m} \cup \{j_m\}$; $c \leftarrow c+1$
4. IF $c = k$ THEN RETURN($I_1, \ldots, I_m$)
5. goto 2.

This method will approach the optimum as we increase m. It is also interesting to have more than one set of colorants to mix, for example one set of colorants may be more stable under small variations on the concentrations than others.

Now we will look at two different functions f or measures on how good a set of colorants is. With them we will try to get as close as possible to a very good set of colorants and we will not treat the more difficult problem of finding a good enough solution.

5.2.1 Minimum distance

Remember from section **??** the geometrical interpretation of equation (**??**): I is very good if the origin is contained in the convex hull H_I of $\{w_i | i \in I\}$. Also we remarked that this will not happen in general, so our aim is that the convex hull is as close as possible to the origin. Thus the Euclidean distance from the convex hull to the origin is the natural way to judge sets I:

$$f(I) = d(0, H_I) = \min\{\|w\| \mid w \in H_I\}. \tag{12}$$

We describe how to calculate this with the following program. By V_K, for any $K \subset I$, we mean the affine space generated by $\{w_i\}_{i \in K}$.

1. $I_1 \leftarrow I; v_1 \leftarrow 0; i \leftarrow 1$
2. Let $\pi_{I_i}(v_i)$ be the orthogonal projection of v_i onto V_{I_i}.

3. IF exists $J_i \subset I_i$ such that $V_{I_i \setminus J_i}$ is an hyperplane in V_i and separates $\{w_j\}_{j \in J_i}$ from $\pi_{I_i}(v_i)$
THEN goto 4
ELSE RETURN($\sqrt{d(v_1, v_2)^2 + \cdots + d(v_{i-1}, v_i)^2}$)

4. $I_{i+1} \leftarrow I_i \setminus J_i$; $v_{i+1} \leftarrow \pi_{I_i}(0)$; $i \leftarrow i + 1$

5. goto 2.

5.2.2 Angle

Let us suppose for a moment that $k = 2$, this means that we have a collection of points $\{w_i\}_{i \in C}$ and our aim is to pick two points w_{i_1}, w_{i_2} such that the interval between both almost contains the origin. If this is the case then the angle $\widehat{w_{i_1} 0 w_{i_2}}$ should be very close to π. And it holds that the angle $\widehat{w_{i_1} 0 w_{i_2}}$ is π if and only if 0 is contained in the interval between w_{i_1} and w_{i_2}. This suggests that the angle might be a good measure. Unfortunately there are very particular cases where this measure is bad. So we will hope that our set of points is general enough and believe that this measure is good.

What happens if $k > 2$? We propose a generalization of angle, namely the solid angle: fraction of the unit sphere overlapped by the cone with the origin as a vertex and generated by the convex hull of our set of points. This number is not easy to calculate unless $k \leq 3$ (for $k = 3$ we have the Gauss-Bonet formula), so the best way to approximate it is by a Montecarlo method. That is, we select random unit vectors uniformly distributed and we count how many lie inside the cone. If we do this for enough vectors we will get a good approximation of the solid angle. For $k > 2$ it is still true that the solid angle of the cone with vertex 0 and generated by $\{w_i\}_{i \in I}$ is $1/2$ of the unit sphere if and only if the convex hull of $\{w_i\}_{i \in I}$ contains the origin.

One further idea is to use as a measure the sum of the angles between all pairs of vectors in $\{w_i\}_{i \in I}$. This works well only if k is small (say $k \leq 6$). For example for $k = 3$ the sum of the angles of all pairs is 2π if and only if the convex hull H_I contains the origin. For $k \leq 4$ there is still a maximum for the sum, in the case when this maximum is attained then the origin is in H_I but the reciprocal does not hold any more.

6 Conclusion

The methods presented in this paper were the result of a week-long brainstorming session. There is nothing final about any of the algorithms we describe. Rather, we show that the hard problem of selecting colorants has a geometrical interpretation that inspires a new kind of strategy to solve the problem.

References

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