Displacement of a viscoplastic fluid in an inclined slot

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Abstract

The steady displacement of one viscoplastic fluid by another is studied in an inclined channel. The aim is a prediction of the finger width from simple balance laws. It is argued that no accurate prediction can be acquired from the far field velocity profiles only, but that instead a calculation of the two-dimensional behaviour near the free interface is needed. It is suggested that the static residual thickness can be determined from a minimization procedure for the dissipation in the system.

Keywords

Viscoplastic, Bingham fluids, Inclined channel, Drilling of oil wells

1 Introduction

The primary cementing operation for the drilling of oil wells consists of the following stages: first drill a new part of the well, trip-out the drill pipe, trip-in the steel casing, pump spacer or lead and tail slurry to displace the drilling mud upwards in an annulus and start again. For the study of the last stage, the miscible displacements of viscoplastic fluids for the purpose of well cementing, a simplified model is discussed of a two-dimensional channel with two Bingham fluids.

Known models for such a displacement of two fluids are based on lubrication type approximations and eccentric annular Hele-Shaw cells, see the references in [2]. During the Study Group attempts have been made (although unsuccessfully) to balance the effect of gravity, say the load of the displacer fluid c, to the lubrication pressure in the thin residual layers, comparable to a lift, using lubrication arguments. In the literature several lubrication scalings have been studied, which indicate that a valid approximation can be made of the velocity field, but not of the yield surfaces. We will not present a discussion of the usefulness of the lubrication models but only refer to the citations in [1],[2].

It seems important to investigate what happens on the gap scale, since numerical and analytical studies indicate that while the front of the displacer fluid moves steadily down the slot, a uniform layer of residual is left behind at the walls. The formation of these static residual layers is observed when the yield stress of the displaced fluid is not exceeded at the walls of the channel. The aim of this study is to predict the thickness of the static residual layers, preferably from a simple criterion and in an inclined channel. In the vertical, symmetric case an accurate criterion for the layer thickness seems to be given by a recirculation criterion. The first aim is therefore to understand the predictive value of the recirculation criterion and to generalize the idea to the non-symmetric case. It is shown in this report why preventing recirculation in the moving frame of reference gives a lower bound for the layer thickness. However, in the case of buoyancy, direct numerical simulations indicate that the recirculation criterion is not accurate and further investigations are required for the flow behaviour near the tip of the interface between the two fluids.

We start in section 2 with a reprise of the rheological properties of the Bingham fluids, the introduction of the dimensionless numbers, and the steady velocity profiles in a channel. In sections 3 and 4 we discuss the recirculation criterion for a vertical and an inclined channel. When a steady numerical calculation in a moving frame of reference is to be done, instead of solving the dynamic system, the difficulty arises of the non-uniqueness of the steady interface between the two fluids (the displacement front). The selection of the finger width and the residual layer thicknesses is likely to be found from a

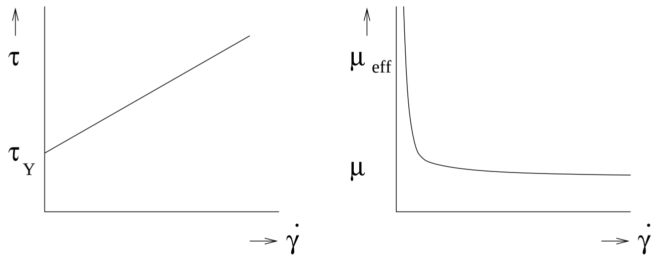

Figure 1: Bingham stress-strain plot and effective viscosity $(\mu + \frac{\tau_Y}{\dot{\gamma}})$.

minimization criterion for the dissipation of energy. In section 5 we discuss the numerical approach using a regularization of both the interface and the rheological properties.

2 Viscoplastic fluids

We will adopt the following notation for the viscoplastic Bingham fluids in a two-dimensional channel.

Suppose fluid m (mud) is being displaced and fluid c (cement slurry) is the displacer. Assuming that both fluids are perfectly Bingham, but with different rheological parameters, the constitutive laws with yield stress τ_Y and viscosity μ (with different values for fluids m and c, in spatial domains Ω_m and Ω_c respectively) are given by:

$$\begin{aligned} \dot{\gamma}(\mathbf{u}) &= 0 && \Longleftrightarrow \quad \tau(\mathbf{u}) \leq \tau_Y \\ \tau_{ij}(\mathbf{u}) &= \left(\mu + \frac{\tau_Y}{\dot{\gamma}(\mathbf{u})}\right) \dot{\gamma}_{ij}(\mathbf{u}) && \Longleftrightarrow \quad \tau(\mathbf{u}) > \tau_Y \end{aligned} \tag{1}$$

where we have used the notation

$$\dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \; \dot{\gamma}(\mathbf{u}) = \left(\frac{1}{2}\sum_{i,j} \dot{\gamma}_{ij}^2(\mathbf{u})\right)^{1/2}, \; \tau(\mathbf{u}) = \left(\frac{1}{2}\sum_{i,j} \tau_{ij}(\mathbf{u})\right)^{1/2}.$$

To avoid viscous fingering the viscosity μ_c of the displacer c is taken to be smaller than μ_m of fluid m; the static layers will typically occur when the yield stress $\tau_{c,Y}$ of fluid c is smaller than that of m.

The equations of motion are made dimensionless relative to the mean displacement velocity U_0, the density of the displaced fluid ρ_m, and the slot half-width D, which gives as dimensionless numbers: the density ratio

$$r = \frac{\rho_c}{\rho_m} \geq 1,$$

the buoyancy parameter

$$b = \frac{(\rho_c - \rho_m)\, gD}{\rho_m U_0^2} \geq 0,$$

and the plastic yield stresses and viscosities

$$\tau_Y = \frac{\tilde{\tau}_Y}{\rho_m U_0^2}, \; \mu = \frac{\tilde{\mu}}{\rho_m U_0 D}.$$

(These should be read with $\tau_{m,Y}$ and $\tau_{c,Y}$, respectively μ_m and μ_c.)

Typically there will be an interface between the two fluids, as indicated in figure 2, that moves steadily along the channel with some speed S. We change to a moving frame of reference in which the shape of the interface is fixed. We choose coordinates (x, y) with corresponding velocities (u, v), with x in axial direction, the y-axis fixed by the tip of the displacement front, and with $\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ denoting

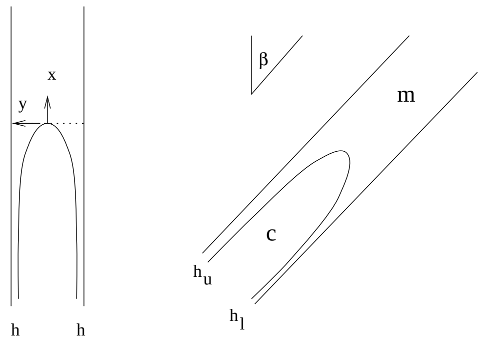

Figure 2: Steady interface between displacer c and displaced fluid m; symmetric and non-symmetric front.

the angle between the x-axis and the vertical, gravitational, direction. The channel thus gives a domain $\Omega = (-L, L) \times (-1, 1)$, where the dimensionless length L is large (but finite), with Ω divided into fluid domains Ω_c and Ω_m separated by an interface Γ. At one end of the channel ($x = L$) there is only the fluid m, and at the other end ($x = -L$) there will be two static layers of fluid m at the walls of the channel, say with thickness h_u and h_l, so that fluid c basically flows though a channel of width $(2 - h_u - h_l)$.

Conservation of volume gives the propagation speed of the front and therefore the proper speed for a moving frame of reference S,

$$S = \frac{2}{2 - (h_u + h_l)}. \tag{2}$$

The ratio of the (dimensionless) yield stress and the viscosity times the squared mean velocity is called the Bingham number of the fluid, so for fluid m, $B_m = \frac{\tau_{m,Y}}{\mu_m}$, and for fluid c, $B_c = \frac{\tau_{c,Y}}{\mu_c S^2}$.

In the moving frame of reference, the continuity and momentum equations read

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

in Ω_c :

$$r\,\mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot \tau - \mathbf{b}, \tag{4}$$

with $\mathbf{b} = b(\cos\beta, \sin\beta)$, and in Ω_m :

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot \tau. \tag{5}$$

Boundary conditions are given by

$$u(x, \pm 1) = -S, \ v(x, \pm 1) = 0,$$

and at the interface Γ the conditions are given by continuity of the velocities and continuity of the normal stresses.

2.1 Far field velocity profiles

The steady profile downstream, where only fluid m is found, is a plane Poiseuille flow of a Bingham fluid; in a moving frame of reference with propagation speed S this is given by

$$u(L, y) = u_L(y) = \begin{cases} \frac{3}{Y_m + 2} - S, & |y| \in [0, Y_m) \\ \frac{3}{Y_m+2}\left(1 - \frac{(|y| - Y_m)^2}{(1-Y_m)^2}\right) - S, & |y| \in [Y_m, 1] \end{cases} \tag{6}$$

where $Y_m = \frac{1}{\xi(B_m)}$, $B_m = \frac{\tau_{m,Y}}{\mu_m}$ and $\xi(B)$ is the only root of the parametric cubic equation

$$2\xi^3 - \left(3 + \frac{6}{B}\right)\xi^2 + 1 = 0 \tag{7}$$

satisfying $\xi(B) > 1$.

Upstream there is, besides the static layers of fluid m at the walls, a plane Poiseuille flow for fluid c,

$$u(-L,y) = u_{-L}(y) = \begin{cases} \frac{3}{Y_c+2Y_i} - S, & |y| \in [0, Y_c) \\ \frac{3}{Y_c+2Y_i}\left(1 - \frac{(|y|-Y_c)^2}{(Y_i-Y_c)^2}\right) - S, & |y| \in [Y_c, Y_i) \\ -S, & |y| \in [Y_i, 1] \end{cases} \tag{8}$$

where $Y_c = \frac{Y_i}{\xi(B_c)}$, with ξ as defined in (7) for $B_c = \frac{\tau_{c,Y}}{\mu_c S^2}$. The position of the interface, at $Y_i = 1-h$, is assumed to be symmetric in (8); without this assumption, we can simply shift the velocity profile $u_{-L}(y)$ over a distance $y_c = \frac{h_l - h_u}{2}$, to obtain two different interface positions at $y = Y_u$ $(= 1 - h_u)$ and at $y = Y_l$ $(= 1 - h_l)$. We remark that due to the scaling, the far field velocity profiles are independent of the buoyancy b (even though the stress and the front speed S are not). This implies that the position of the finger of displacer fluid c (in the y-direction) is in a way independent of the buoyancy, or in other words, the thickness of the residual layers of fluid m are to be determined solely by the dynamics around the tip of the finger.

For the existence of the static residual layers, it is necessary that the shear stress at the wall of the channel does not exceed the yield stress of fluid m. For example for the symmetric case, this translates in a maximal value of the thickness, $h_{\max}$, determined by the condition

$$\tau_{wall} = \frac{\tau_{c,Y}\xi(B_c)}{Y_i} + b(1 - Y_i) \leq \tau_{m,Y}. \tag{9}$$

This means that $h_{\max}$ is defined by $1 - \frac{1}{S_{\max}}$ with $S_{\max}$ the velocity at which equality is attained in (9), with both B_c and Y_i dependent on S. We see from this that the condition $\tau_{c,Y} > \tau_{m,Y}$ is indeed a necessary condition to find a $h_{\max} > 0$.

3 Recirculation criterion

In this section we discuss the symmetric case, in a vertical channel.

Apart from the maximal layer thickness $h_{\max}$, as determined by (9), another limit for the static layer thickness can be found by looking at the velocities at the centerline at $y = 0$. Since the mean velocity upstream at $x = -L$ is given by the speed S, or actually by 0 in the moving frame, the maximum speed at $y = 0$ has to exceed this, so $u_{-L}(0) > 0$, and the velocity has to drop down to zero when we are moving along the centerline towards the tip of the finger. Since the displacer fluid is pushing the fluid m out of the channel, it can be expected that the maximum velocity will decrease further, so that $u_{-L}(0) > 0 \geq u_L(0)$. Here pushing is seen as a compressive stress, with $u_x \leq 0$, while an increase in velocity, $u_x \geq 0$ would imply that the fluid m is pulling the displacer c out of the channel. The inequality $0 \geq u_L(0)$ predicts that there will be no recirculation downstream (in the moving frame) and translates into a critical value of the layer thickness. This value for h at which $0 = u_L(0)$ is denoted by h_{circ} and it gives a lower bound for the speed S; from (6),

$$S \geq \frac{3}{1/\xi(B_m) + 2} \; (= \frac{1}{1 - h_{circ}} = S_{circ}).$$

It is observed in numerical calculations of the displacement in a vertical channel, that h_{circ} is in fact a rather good predictor for the value of the actual thickness h, i.e. equality is nearly attained. How can this be understood? Consider the situation when there is only a small compressive stress in fluid m, which means that at the centerline $y = 0$ the total stress $|\tau|$ is certainly smaller than $\tau_{m,Y}$. Along the centerline, where also the shear stress vanishes, this results in a rigid movement and the velocity has to be equal to the velocity at the interface. It can thus be expected that when the yield stress of m is sufficiently large, the recirculation criterion is very accurate. However, if the yield stress $\tau_{m,Y}$ is close to that of fluid c, it should be expected that the speed S is larger than the critical recirculation speed. Indeed, since the interface causes a fully two-dimensional flow, there have to be compressive stresses τ_{xx}

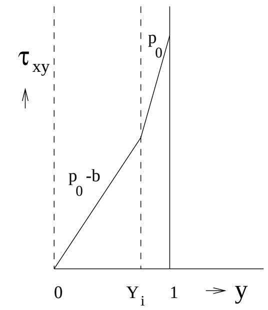

Figure 3: Shear stress in the two Bingham fluids, upstream at $x = -L$.

and τ_{yy} in fluid c, that exceed the yield stress $\tau_{c,Y}$ (since $\dot{\gamma} \neq 0$ but $\tau_{xy} = 0$ along $y = 0$) and since the normal stresses are continuous over the interface, they will exceed the yield stress of m and the velocity decreases further along the centerline. Similarly, if the viscosity of m is relatively large, the stresses near the tip will be big enough to "melt" the fluid m at the centerline, and the speed deviates from the recirculation speed. These effects are not all that clear in the data in [2] where comparisons of h and h_{circ} are shown using variations in the rheological parameters (figures 15 and 16 in [2]). The argument does give an idea, however, why the layer thickness decreases (with a decrease in S) with increasing yield stress or decreasing viscosity of fluid m, as is observed in the numerical computations.

We remark that the value of h_{circ} is fully determined by the far field conditions, which are independent of the buoyancy b. When looking at the data of the layer thickness for different values of the buoyancy b, see figure 4 below for $\beta = 0$, we observe that the actual layer thickness is larger than predicted by the recirculation criterion (as can be expected from the argument above), but furthermore that it varies with b.

We expect that for larger values of b the recirculation criterion becomes more accurate (see also figure 4 for $\beta = 0$). In figure 3 a plot of the shear stress is shown in the far field with residual layers. The slope of the stress in the center part is given by the modified pressure $\tilde{p}_0 = p_0 - b = \tau_{c,Y} S\xi(B_c)$, where p_0 denotes the pressure gradient in the channel that is applied to achieve a throughput of 2 (after scaling). This means that in the static layers of fluid m, the slope of the shear stress is given by $\tilde{p}_0 + b$. The maximal layer thickness $h_{\max}$ is determined by the shear stress at the wall, as given in (9). From the figure we can conclude that when b is increased substantially, τ_{wall} will increase and h will need to decrease, since a static layer can only exist if h does not exceed $h_{\max}$; but there is a lower bound for h given by h_{circ}. It is therefore expected that for large b, the front speed S will approach S_{circ}. This is related to the idea that if the material that is pushed away is very light (b large), then it is simply pushed away without high stresses, which means that the yield stress is not exceeded at the centerline in the light fluid, therefore, by the argument given above, $S = S_{circ}$.

4 Inclined channel

When we consider the case of an inclined channel at angle β, with buoyancy parameter b, the same argument as before can be given for a lower bound of the front speed S. Suppose that the tip of the finger (where the speed vanishes exactly) lies within the y-interval $[-Y_m, Y_m]$, where the fluid m downstream behaves like a solid, then $S \geq S_{circ}$ as before. Note that at any point on the interface away from the tip, the velocity u will be smaller, so that when the tip does not lie within the solid-interval, the estimate will be less sharp.

It should be noted again that the recirculation criterion contains neither buoyancy or inclination angle and indeed, the criterion does not give an accurate correspondence with numerical results in an inclined channel. We use the numerical data provided in [2]. Some simple data fitting indicates that the plots may be linear in $\sin\beta$ and $\cos\beta$, but they are not linear in the buoyancy parameter b. The data also seem to indicate that both the total finger width $2 - (h_l + h_u)$ and the position of the centerline of fluid c, $(h_l - h_u)/2$, are monotonic in the buoyancy b and the inclination angle β.

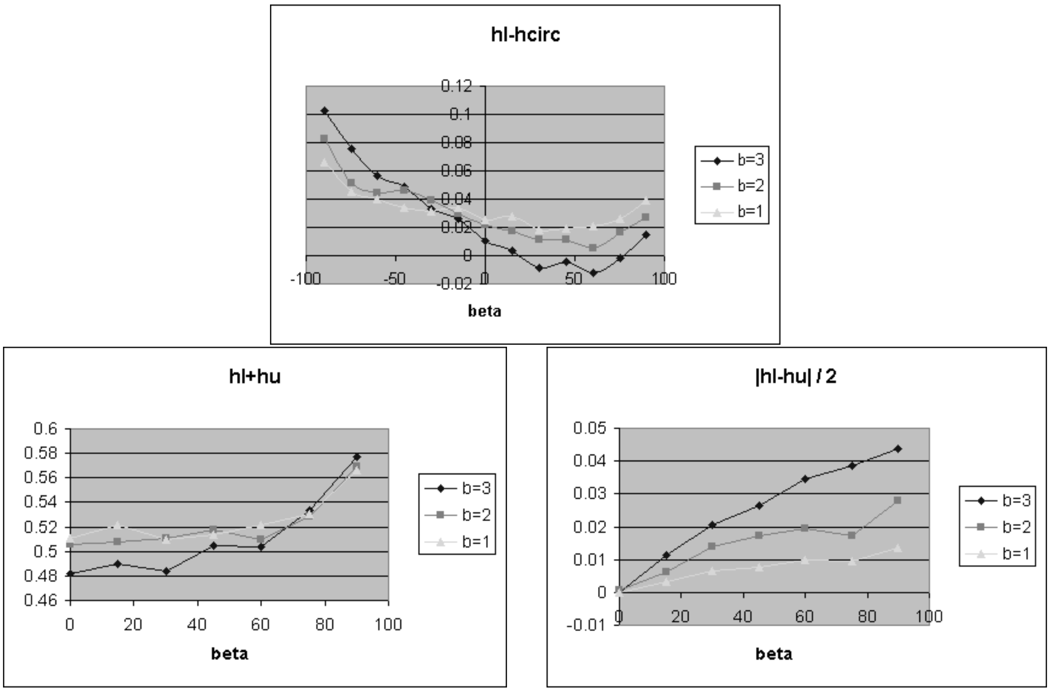

Figure 4: Thickness of the residual layer $(h_l - h_{circ})$ for inclination angle β; the lower two figures show the total layer thickness and the centerline position y_c.

Since the layer thicknesses, upper and lower, depend on the buoyancy and inclination angle, while the volume conservation (2) and recirculation criterion do not, additional information should be gained from the momentum equation. Unfortunately, the momentum equation cannot be integrated over the far field profiles only, since most of the viscous dissipation occurs near the front of the interface. The incompressibility of the two-dimensional material allows one to visualize the flow using the contourlines of the Stokes streamfunction, defined by $u = \psi_y, v = -\psi_x$. The fact that there is no recirculation in the moving frame, implies that the contourlines of the streamfunction do not have large gradients, which can be interpreted as a minimization of viscous dissipation.

If the inner product is taken of velocity $\mathbf{u}$ and (4) and (5), and integrated over Ω, ignoring inertia effect for simplicity, we observe

$$\oint_{\partial\Omega} -p\mathbf{u}\cdot\mathbf{n} + \int_{\Omega} \tau : \nabla\mathbf{u} + \int_{\Omega_c} bv\sin\beta = 0. \tag{10}$$

Here we have used that over the interface, the pressure, the normal stress and the velocities are continuous, so that contributions from the two fluids balance:

$$\oint_{\Gamma} -p\mathbf{u}\cdot\mathbf{n} = \int_{\Gamma} (\tau\cdot\mathbf{n})\cdot\mathbf{u} = 0,$$

while at the outer boundaries the no-slip condition gives

$$\int_{\partial\Omega} (\tau\cdot\mathbf{n})\cdot\mathbf{u} = 0,$$

and, finally, using that the interface Γ is a streamline, which can been defined as the contour for $\psi = 0$, we find that

$$\int_{\Omega_c} bu\cos\beta = \int_{-L}^{0} b\psi\cos\beta]_{y=-Y_i(x)}^{y=Y_i(x)}\, dx = 0.$$

Again using the no-slip condition at the channel walls, the first integral in (10) reduces to contributions from the far field, at $x = -L$ and $x = L$, where the pressure and velocity are known explicitly, given by the Poiseuille flow in (6) and (8). Writing for the pressure gradient along the channel p_0, we find that near $x = -L$,

$$p(x, y) = -p_0 x - b\sin\beta(y - 1 + h_u),$$

while towards $x = L$ simply

$$p(x, y) = -p_0 x.$$

This means that the first integral in (10) is given by $4p_0 L + 2b\sin\beta(1 - \frac{h_l - h_u}{2})$. The third integral can also be calculated using the expression for the pressure, using that

$$\int_{\Omega_c} bv\sin\beta = -\int_{-Y_i}^{Y_i} b\psi\sin\beta]_{x=-L}^{x=Y_i(y)}\, dy = \int_{-Y_i}^{Y_i} pyu\sin\beta\, dy = 2b\sin\beta\,\frac{h_l - h_u}{2}.$$

We thus conclude that the viscous dissipation is given by

$$\int_{\Omega} \tau : \nabla\mathbf{u} = -2\left(2p_0 L + b\sin\beta(1 - h_u)\right), \qquad (11)$$

where the last term can be written using S in (2). Observe that this last term does not depend on $2L$, the length of the channel under consideration, therefore this dissipation takes place in a localised region near the tip of the finger. This dissipation can not be determined without explicit knowledge of the shape of the free interface and the corresponding flow near this interface and therefore it cannot be expected to supply a simple integral criterion for the flow characteristics. Both the interface and the stresses near this interface need to be calculated in a dynamical simulation or other criteria have to be found from a numerical calculation of the steady case.

Dynamical calculations have been done by Schlumberger previously [2], based on the dynamic problem with fully two-dimensional displacement computations in a volume-of-fluid-method. Most of the computations were done with rheological parameters: $(b, \tau_{c,Y}, \tau_{m,Y}, \mu_c, \mu_m) = (0, 0.2, 0.5, 0.01, 0.05)$, and $(0, 0.2, 1.0, 0.005, 0.01)$, for which $h_{circ} = 0.13$ and 0.04 respectively. The data shown in figure 4 below, however, are from calculations in an inclined channel with $h_{circ} = 0.23$. Steady calculations have not been done, but we would like to make some remarks about the difficulties with such computations.

5 Numerical approach

It is shown in [1] that there does not exist a unique solution for the steady interface, in the sense that given an interface, sufficiently smooth, a solution of the velocities or streamfunction can be found in the two fluid domains Ω_c and Ω_m. This implies that the selection of the interface is a dynamic effect or that it is selected by a minimization principle, for instance by minimization of viscous dissipation. The recirculation criterion and the contourplot of the streamfunction seem to indicate that the viscous dissipation is a good selection criterion, but numerical calculations should substantiate or gainsay such a claim.

The numerical approach is to do a regularized problem, without a sharp interface and by smoothing out the rheological properties. The interface regularization can be done by modelling the displacement as the advection of a passive scalar, say a concentration C; this is achieved by replacing the kinematic condition at the interface $y = Y_i(x, t)$ by the advection

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = 0$$

with $C = 1$ in Ω_c and $C = 0$ in Ω_m. This gives a diffuse region of intermediate concentration instead of a sharp interface. The intermediate concentration values are only observed in a thin region determining the smoothened interface; for any meshpoint in this region the concentration-dependent rheology can be used, for example $\mu(C) = C\mu_c + (1 - C)\mu_m$.

The effect of the smooth interface is that the displacement is considered as a flow of only one fluid, with different rheological properties at different positions. Another way to dismiss the interface by considering the fluids to be effectively one fluid, is to fix the rheological parameters as a function of position determined by the zero contourline of the previous iterate for the streamfunction ψ.

The effective viscosity of this fluid is regularized by using

$$\mu_{eff} = \mu + \frac{\tau_Y}{\dot{\gamma}(\mathbf{u}) + \varepsilon}$$

with ε a fixed, small parameter, and all the other parameters depending on the concentration C or the position.

In a streamfunction formulation, the steady problem is now reduced to a fourth order problem in ψ, with only boundary conditions to be provided at the boundary of the specified domain Ω (so without interface). The conditions at the channels walls are the no-slip conditions, $u = -S, v = 0$, and at the in- and outlet the conditions are dictated by the far field velocity profiles. The only unknowns in these conditions are the layer thicknesses h_u and h_l, which will be the parameters over which the dissipation is minimized, for example in a steepest descent approach. In the streamfunction formulation, the thicknesses are determined by the distance of the zero contourline from the wall; in the case where the interface is regularised with a concentration C, the position of the interface should be found from interpolation. We do not expect these steady calculations to be more cost efficient than the dynamic problem, but it may provide insight in the selection of the front and the corresponding layers.

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