

Chapter 4

On Seismic Imaging: Geodesics, Isochrons, and Fermat's Principle

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4.1 Introduction

The problem described in this report deals with the propagation of waves in elastic solids. The particular application was suggested by Baker Atlas, PanCanadian Petroleum, Petro-Canada Oil and Gas, and Talisman Energy, i.e., the companies involved in seismic data acquisition, processing and interpretation and sponsoring The Geomechanics Project at The University of Calgary.

Seismic data is used to obtain the information about the subsurface, which includes spatial positions of reflectors. In the case of exploration seismology, the known parameters are: the location of the source, the location of the receiver, and traveltime of a signal between the source and receiver. (In earthquake seismology the first and the third parameter might have to be inferred.)

To proceed with the geophysical investigations one often needs to assume knowledge of the material field which describes the velocity of a signal as a function of position (inhomogeneity) and direction (anisotropy). Consequently, fitting the observed data with the assumed characteristics of the material field leads to an image of the subsurface. We investigate the scattering from a subsurface point of a signal generated by the source and recorded at the receiver. If the signal velocity in the medium is constant, i.e., the medium is homogeneous and isotropic, the set of all possible reflection points for the measured traveltime forms an ellipse whose foci correspond to the source and receiver positions and whose "string length" is equal to the product of the assumed velocity and the measured traveltime. This ellipse is an isochron, the locus of the same traveltime, corresponding to all possible reflection points of the traveltime geodesics. (The geodesics are the least-time raypaths between a given source and receiver.) A continuous set of reflection points, such as a planar reflector, corresponds to the reflective surface whose location coincides with intersection of numerous ellipses generated by numerous source-receiver configurations. The location of the intersection points also coincides with the outer envelope (convex hull) containing all these ellipses.

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The intersection points have served as a standard way for creating a subsurface image based on seismic data derived from numerous sources and receivers.

A more accurate model of the material field should take into account both inhomogeneity and anisotropy of the subsurface. Consequently, the geodesics might not form straight lines and isochrons might not form ellipses. Furthermore, the intersection points might not coincide with the convex hull. For instance, complex material fields required for accurate imaging of seismic data acquired in the foothills might not allow the benefit of certain simplifying assumptions. Notably, the convenient coincidence of intersection points and the envelope should be revisited. In order to provide a more general approach we allow sources and receivers to be located at different levels, for instance in data acquisition over a complicated surface, or positioning of receivers in the wellbore, as in Vertical Seismic Profiling.

This report constitutes a significant contribution of all the authors. In particular, the selected sections are derived from written documents provided by A.T. Dawes, A.M. Duffy, A.J. Irwin, S.T. Jensen, and E. Nyland. The aforementioned documents are available upon request. Also, C.S. Bohun, E. Nyland and P.S. Webster wrote computer code in Mathematica and Maple, which is available on request.

4.2 Background Of The Problem

4.2.1 Mathematical foundations

The differential equations that describe the propagation of seismic pulses in a heterogeneous anisotropic material are well known [3]. If a continuum with elasticity tensor C_{ijpq} and mass density ρ undergoes a space and time dependent deformation u_i and the spatial derivatives of the displacement are small, then

$$(C_{ijpq}u_{p,q})_{,j} = \rho\ddot{u}_i.$$

Suppose this equation has a solution of the form

$$u_i(\mathbf{r}, t) = U_i(t - T(\mathbf{r}))f(\mathbf{r})$$

where U_i is essentially zero unless the time t is near the traveltime, $T(\mathbf{r})$, required for a wavefront to travel from the origin of coordinates to position \mathbf{r} . The scalar function f is smooth. Clearly $U_i \ll \dot{U}_i \ll \ddot{U}_i$ and all derivatives of f are also small. By ignoring all terms involving first or no derivatives of U and f , this elastodynamic equation can be written as

$$(\rho\delta_{ik} - C_{ijkl}T_{,j}T_{,i})\ddot{U}_k f \approx 0 \implies |\rho\delta_{ik} - C_{ijkl}T_{,j}T_{,i}| = 0.$$

In the case of isotropic media the last equation becomes

$$(\nabla T \bullet \nabla T - 1/\alpha^2)(\nabla T \bullet \nabla T - 1/\beta^2)^2 = 0$$

where α and β are the velocities of propagation of compressional and torsional pulse fronts respectively. The surface $T = \text{constant}$ defines the pulse front and these eikonal equations imply that the perpendiculars to this pulse front, the ray trajectories, must satisfy the parametric equations

$$\partial_s \frac{\partial_s \mathbf{r}(s)}{c(\mathbf{r})} = \nabla \frac{1}{c(\mathbf{r})}$$

where c is either α or β and the solution is sought in a domain where the velocity is essentially constant over the width of the pulse. These equations for the ray trajectory are equivalent to Fermat's principle of stationary time along the trajectory

$$\delta \oint_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{1}{c(\mathbf{r})} ds = 0 \iff \partial_s \frac{\partial_s \mathbf{r}(s)}{c(\mathbf{r})} = \nabla \frac{1}{c(\mathbf{r})}$$

4.2.2 A physical model of propagation of a seismic pulse

This mathematical development is a strict definition of a model for wave propagation first proposed by Christian Huygens in the 17th century. It is possible to take the view that a propagating disturbance has a well-defined wave front at any time t and that it interacts with the medium only on the wave front. At every such point in the medium, energy is scattered in all directions. After an instant the new wave front is the envelope of all such scattering “wavelets”, Huygens wavelets.

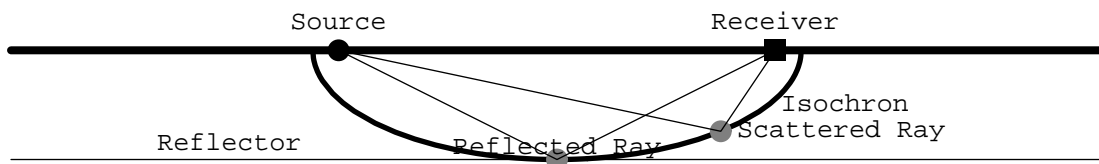
In seismology, a disturbance that travels in a fixed time from a source to a receiver could have been scattered as a Huygens wave from a scatterer that can have a multitude of positions. The collection of scatterers that lead to constant travel time are called an isochron. For travel in three dimensions, the isochron is a surface while in two dimensions the isochron is a curve.

Once the speed is determined as a function of position and direction, this scattering isochron can be calculated. In a medium of constant velocity the scattering isochron is obviously an ellipse with the source of the energy at one focus and the receiver at another. These isochrons are more complicated if the velocity varies with position or the elasticity tensor is anisotropic. If isochrons corresponding to different source receiver pairs intersect, the intersection points define the times at which energy on the two records is scattered from the same point. If the scattering is from a reflector a little thought will show that the reflector must be the envelope of all the scattering isochrons that include a point on the reflector.

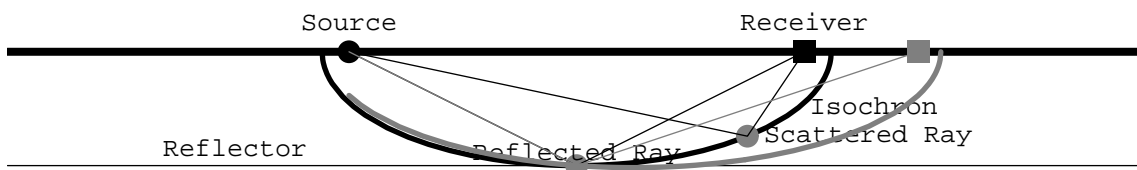
4.2.3 Single horizontal reflector

In order to clarify the concepts, consider the case when a single, horizontal, reflector exists within a medium in which velocity is generally a slowly varying function of position. Although the isochrons are not exactly ellipses it will be convenient to draw them as such.

The seismologist does not know of the existence of the reflector before doing measurements. If scattered energy is observed in a detector at time T , after the source was initiated it is possible that this energy was scattered from any point in the medium that lies on the isochron. If the isochron includes the actual reflection point the geometrical relations are easy to illustrate



but the seismic record at the receiver usually has signals that arrived at other times as well. Such signals were scattered from points defined by other isochrons. Consider now the effect of moving the detector of energy along the surface.



The isochrons for the second detector position generally intersect those generated for the first. In particular, one of the new isochrons intersects the reflection point for the first position and the energy seen at the time corresponding to the new receiver position could therefore have been scattered in part from the same point on the reflecting horizon. Since the signals from the same scattering point might reasonably be expected to be in phase, the scattered signal from this reflection point could be enhanced by “stacking” the two portions of the response together. It is elementary to show that for

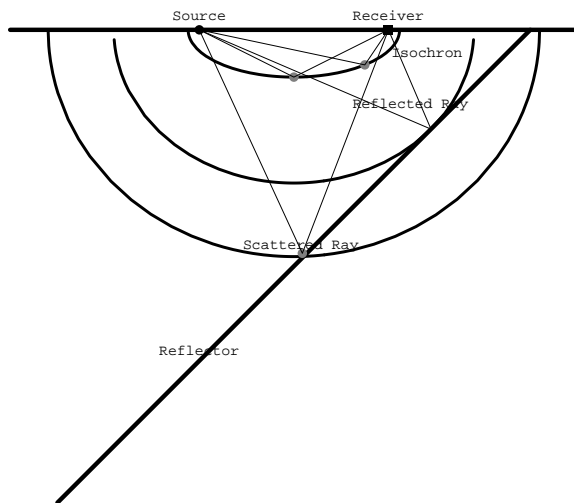
a layer of constant velocity c , the arrival time for energy scattered from a reflecting point at offset X from the source at the origin of coordinates is determined by

$$T^2 - \frac{x^2}{c^2} + 2\frac{xX}{c^2} = 2\frac{d^2 + X^2}{c^2}$$

Notice that the minimum travel time is at $x = X$, as it ought to be. When considered on a plot of signal as a function of detector position x and time along the recording T , this stacking curve is a hyperbola with its apex directly below the position of the reflecting point.

4.2.4 A single dipping reflector

The geometry changes if the reflector dips, or if the detector positions are on the side of a hill and the reflector is flat. Find the points at which the isochron intersects the reflector as a function of the path length. If these points coalesce you have the reflection point.



For a constant velocity layer the isochrons are still ellipses with foci at the source and receiver positions, but different isochrons now contain the reflection point. The nature of the stacking curve remains the same though because scattering from a point is assumed to depend only on the position of the scattering point, not the slope of the reflector. The equation of the scattering hyperbola is now

$$T^2 - \frac{x^2}{c^2} + 2\frac{xX}{c^2} = 2\frac{((d_0 - x \tan(\theta))^2 + X^2)}{c^2}$$

and the horizontal position at which the scattering path is a minimum is now

$$x = \frac{X + 2d_0 \tan(\theta)}{1 + 2 \tan(\theta)}$$

4.2.5 Vertical seismic profiling

It is much easier to place detectors on the surface of the earth, but if a well exists in the neighborhood of the target for the seismic investigation, it is perfectly reasonable to use detectors inside the well to measure travel times closer to the reflector. The isochrons are still ellipses but now one of the foci is on the surface while the other is beneath it. The stacking curves are now far from simple, particularly if the offset of the source from the well is significant. It is this problem that was explored at PIMSIPS 98.

4.3 Theoretical Approach

4.3.1 Constant velocity field

For a constant velocity field $v(z) = A$, we can formulate the Lagrangian of the total time taken by the acoustic wave to travel from source to receiver as

$$\begin{aligned}\mathcal{L} &= \int_a^0 \frac{\sqrt{1 + \dot{z}^2}}{v(z)} dx \\ &= \int_a^0 \frac{\sqrt{1 + \dot{z}^2}}{A} dx.\end{aligned}$$

The Euler-Lagrange equation is employed to minimize this Lagrangian function, yielding

$$\frac{\partial}{\partial x} \left(\dot{z} \frac{\partial \mathcal{L}}{\partial \dot{z}} - \mathcal{L} \right) = 0,$$

$$\frac{\partial}{\partial x} \left(\frac{-1}{v \sqrt{1 + \dot{z}^2}} \right) = 0.$$

The vanishing derivative of this expression implies that this expression is constant, thus

$$E = \frac{-1}{v \sqrt{1 + \dot{z}^2}} \Rightarrow \dot{z} = \sqrt{\frac{1 - v^2 E^2}{v^2 E^2}} \Rightarrow \int_a^x \sqrt{\frac{1 - v^2 E^2}{v^2 E^2}} dx = \int_{z(x)}^{z(a)} dz.$$

Evaluating at the point $(0, d)$ we have

$$E^2 v^2 = \frac{a^2}{a^2 + d^2}.$$

Our integral, with this substitution, is then evaluated as

$$\int_a^x \frac{d}{a} dx = \int_{z(x)}^{z(a)} dz \Rightarrow \frac{z(x) - z(a)}{x - a} = \frac{d}{a}.$$

This result confirms that the geodesic curve for a constant velocity field is a straight line. We can now assume that an acoustic wave that is reflected once within the medium will follow a path consisting of two straight lines, first from source to point of reflection, and then from point of reflection to receiver:

$$l_1 = \sqrt{(a - x)^2 + z^2}, \quad l_2 = \sqrt{x^2 + (d - z)^2}$$

The minimum time taken is therefore the sum of the time taken along both of these lines.

$$v T_{total} = l_1 + l_2,$$

$$v T_{total} = \sqrt{(a - x)^2 + z^2} + \sqrt{x^2 + (d - z)^2}.$$

For given a, d, v, T_{total} , this relationship forms an ellipse on the (x, z) -coordinate axis. By performing many experiments with different values of a and d , corresponding to different positions of the emitter and detector, we will get a range of values for T_{total} . We can then plot the family of ellipses arising from these data points and the profile of the reflective surface will be indicated by where short arcs of the ellipses are concentrated.

4.3.2 Linear velocity field

Determination of direct signal path

The direct signal, which travels from the source to detector in the least amount of time, will follow a path which can be determined by minimizing the functional which gives travel time as a function of path. In the case where the velocity field varies only with z , the problem is effectively two-dimensional. That is to say, if the signal starts in the (x, z) -plane, the path of the direct signal will not deviate from this plane. In this case the travel time functional as a function of path is:

$$T[x(z)] = \int_0^Z F(z, x, x') dz = \int_0^Z \frac{\sqrt{1 + x'(z)^2}}{a + bz} dz, \quad (3.1)$$

where the path is described by $x(z)$ and the integration is from the surface at $z = 0$ to the depth of the detector at $z = Z$. The Euler-Lagrange equation for a functional of this form is:

$$\frac{\partial}{\partial x} F(z, x, x') - \frac{d}{dz} \left(\frac{\partial}{\partial x'} F(z, x, x') \right) = 0. \quad (3.2)$$

In the case that $F(z, x, x')$ does not depend on x this simplifies to:

$$\frac{d}{dz} \left(\frac{\partial}{\partial x'} \frac{\sqrt{1 + x'(z)^2}}{a + bz} \right) = 0, \quad (3.3)$$

or:

$$\frac{\partial}{\partial x'} \frac{\sqrt{1 + x'(z)^2}}{a + bz} = \frac{x'(z)}{(a + bz)\sqrt{1 + x'(z)^2}} = \text{constant}, \quad (3.4)$$

with boundary conditions determined by the source and detector configuration. The boundary conditions are simply that the signal starts from the source at $(X, 0)$ and is received by the detector at $(Z, 0)$. The first boundary condition is $x(0) = X$, since at the surface (where $z = 0$) the signal starts at the source which is a distance X from the detector well. The second boundary is $x(Z) = 0$, since the signal is received by the detector at depth $z = Z$ with no displacement from well, $x = 0$. The above differential equation may be solved to yield:

$$x(z) = \frac{X}{2} - \frac{Z(2a + bZ)}{2bX} + \sqrt{\left(\frac{X}{2} + \frac{Z(2a + bZ)}{2bX} \right)^2 - \frac{z(2a + bz)}{b}} \quad (3.5)$$

which describes the path of the direct signal for the prescribed velocity field, and represents an arc of a circle.

4.3.3 Direct signal travel time

In order to calculate the travel time of the direct signal, the solution must be differentiated with respect to z and substituted into the travel time functional. The differentiation of equation 3.5 yields:

$$\frac{\partial}{\partial z} x(z) = - \left(\frac{a + bz}{b} \right) \left(\sqrt{\left(\frac{X}{2} + \frac{Z(2a + bZ)}{2bX} \right)^2 - \frac{z(2a + bz)}{b}} \right)^{-1}. \quad (3.6)$$

This is substituted into the integrand of equation 3.1 and the resulting expression is integrated along the ray path from source to detector to yield the time for the direct signal to arrive as a function of the X , and Z for given constants a and b . The integrand does not converge uniformly as X goes to zero, thus the case $X = 0$ must be treated separately. The following is valid for $X = 0$:

$$\begin{aligned} T(X, Z) &= \int_0^Z \frac{1}{a + bz} dz \\ T(X, Z) &= \frac{\log a + bZ}{b} - \frac{\log a}{b}. \end{aligned} \quad (3.7)$$

as for the case $X \neq 0$ (henceforth this will be the case, unless stated otherwise) we get:

$$\begin{aligned}
 T(X, Z) &= \int_0^Z \frac{1}{a+bz} \sqrt{1 + \left(\frac{a+bz}{b}\right)^2 \left(\left(\frac{X}{2} + \frac{Z(2a+bZ)}{2bX}\right)^2 - \frac{z(2a+bz)}{b}\right)^{-1}} dz \\
 T(X, Z) &= \frac{1}{b} \tanh^{-1} \left(\frac{\sqrt{(Z^2 + X^2)(4a^2 + 4abZ + b^2Z^2 + b^2X^2)}}{|2aZ + bZ^2 + bX^2|} \right). \tag{3.8}
 \end{aligned}$$

Thus for different sets of values for X and Z and measured direct signal times T , one could solve the resulting system of equations for a and b .

The previous equation can then be rewritten as:

$$T(X, Z) = \frac{1}{b} \tanh^{-1} \left(\frac{\sqrt{(Z^2 + X^2)(Z^2 + 4Za/b + 4a^2/b^2 + X^2)}}{|2Za/b + X^2 + Z^2|} \right),$$

which may be solved for b to yield:

$$b = \frac{1}{T(X, Z)} \tanh^{-1} \left(\frac{\sqrt{(Z^2 + X^2)(X^2 + Z^2 + 4Za/b + 4a^2/b^2)}}{|2Za/b + X^2 + Z^2|} \right)$$

Thus it is possible to determine both a and b .

4.4 Numerical Approach

In order to be able to generate pictures showing how the isochrons look for various velocity functions, the Maple symbolic manipulation package was used to perform the calculations.

Firstly, the positions of the sources and receivers was given and then a reflective surface was assumed, so the program could calculate the travel times to those detectors from the sources.

We imagine that the *only* information we have are the travel times, and positions of the sources and receivers as would be the case for a geophysicist. What can we say about the reflective surface that gave these travel times?

We know the reflection (supposing there to be only one reflection) must come from some point of the isochron for that travelttime for the given receiver and source. Maple calculates what the isochron looks like for each travelttime in a given velocity field and shows all the isochrons together.

Once we have all the isochrons shown, we can also display on the same diagram the original reflective surface and from this determine any relation between the two.

4.4.1 Method of solution

The equation for the reflector was taken to be a straight line given by the expression $z = 1500 - x$. The position of the source was at $z = 0$, $x = 635m$ and the receivers were taken to be at $x = 0$ and $z = 1200, 1250, 1300, 1350$ and $1400m$. The specific representative velocity profiles used were

$$v(z, \dot{z}) = \begin{cases} 2315 & \text{A: isotropic} \\ 2315 \left(1 + \frac{0.5}{1 + \dot{z}^2}\right) & \text{B: anisotropic} \\ 2315 + 0.8z & \text{C: linear} \end{cases}$$

Isochronous curves for the three separate velocity profiles were generated using the Maple symbolic manipulation package. For a given profile the geodesics are determined by computing the class of curves C that minimize the total time of traversal

$$T = \int_C L(z, \dot{z}) d\tau = \int_C \frac{(1 + \dot{z}^2)}{v(z, \dot{z})} d\tau.$$

The curve C joins two points (x_0, z_0) and (x_1, z_1) and is describing using the parameter τ . Since $L(z, \dot{z})$ is independent of x , the Euler–Lagrange equation becomes

$$\frac{d}{dx} \left(\dot{z} \frac{\partial L}{\partial \dot{z}} - L \right) = 0.$$

In both the isotropic and anisotropic cases the geodesics are straight lines whereas in the case of the linear velocity profile, they are circular arcs. For each case, a parametric form of the geodesics is entered externally into the model. Once the geodesics are known, an expression is found for the total traveltime of traversal from the source to each of the five receivers where the condition that the signal hits the reflector only once is imposed. By parameterizing the equation for the reflector, local minima in the total travel time are found. The location on the reflector of these local minima correspond to the reflection point on the reflector.

To get the corresponding isochronous curves, one again considers the expression for the total traveltime where the point of reflection on the reflector is allowed to vary and the traveltime is the value determined above with the minimization procedure. A contour plot of this two parameter surfaces generates the isochronous curves. Isochronous curves for the three velocity profiles discussed, with a linear reflector, are plotted below.

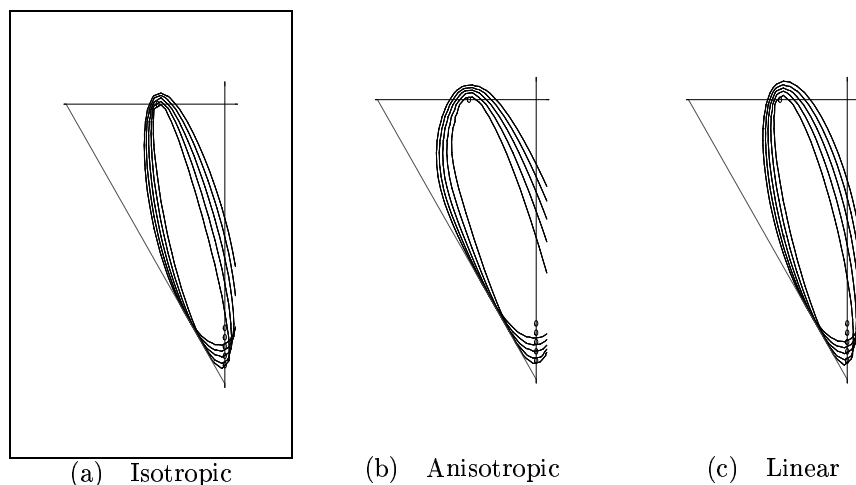


Figure 4.1: Isochronous curves for the three profiles. These three plots show the isochronous curves and illustrate how the envelope of the curves reproduce the location of the reflector, for three different velocity profiles. For all cases the source is indicated at 635m and the five receivers are at depths of 1200, 1250, 1300, 1350 and 1400m.

Clearly this technique can be extended to more complicated velocity profiles such as combining linear velocity dependence with angular dependence.

Lastly, it should be pointed out that for *isotropic* media (i.e. with no angular velocity dependence) the convex hull of the isochrons *must* be the reflective surface (for many isochrons). This is because for this type of media the angle of reflection is equal to the angle of incidence and can thus be considered to reflect off a line that is *perpendicular* to the normal defined by these angles. But this line is also the tangent to the isochron and thus the reflective surface must be along the collection of tangents of all the isochrons. This is the definition of a convex hull.

4.5 Conclusions

Both the greater accuracy of data acquisition/processing and increased scarcity of petroleum resources promote and necessitate rigorous investigations. Consequently, we have attempted to remove certain simplifying assumptions in order to increase the reliability of the seismic image. Firstly, we

Receiver	Depth of Receiver	Velocity profile	Contact position		Travel time (<i>ms</i>)
			x	z	
1	1200	A	386.3	1113.7	663.9
		B	499.2	1000.8	594.6
		C	380	1120	539.0
2	1250	A	336.3	1163.7	669.0
		B	425.4	1074.6	603.7
		C	335	1165	541.9
3	1300	A	281.7	1218.3	674.6
		B	348.9	1151.1	613.0
		C	285	1215	545.4
4	1350	A	221.7	1278.3	681.0
		B	270.0	1230.0	622.7
		C	230	1270	549.5
5	1400	A	186.8	1344.6	687.9
		B	499.2	1313.2	632.7
		C	168	1332	554.4

Table 4.1: Summary of numerical results.

The three velocity profiles studied are A: isotropic, B: anisotropic and C: linear. The depth of the receivers and the contact position with the reflector are given in meters whereas the travel time of the signal is given in milliseconds.

have derived analytic expressions for the isochrons in an inhomogeneous medium characterized the constant vertical velocity gradient, and for the anisotropic medium characterized by the elliptical velocity dependence. The derived algorithms were coded and hence numerical as well as graphical results were generated. The results were generated for a single surface source and five wellbore receivers over a dipping interface. Three types of media were assumed: homogeneous/isotropic, homogeneous/anisotropic, and linearly inhomogeneous/isotropic. The traveltimes and reflection-point results appear to be sensitive to a given assumption of the velocity field. In considering complex geological areas, e.g., foothills, one should examine several velocity fields used in data processing and quantify the difference in derived location of reflection points (e.g., see Table 4.1). Moreover, in such an area, the location of the planar reflector is tangent to the convex hull of isochrons and might not correspond to their intersection points since the two concepts need not coincide. Note that the latter method is commonly used in the Kirchhoff migration procedures.

The immediate further work appears to be the derivation of analytic expressions or numerical algorithms providing the range of positions of the subsurface features depending on assumed velocity fields. Notably, such a quantitative tool would play an important role in the drilling process. Also, the indication of likely scope of displacement in seismic mapping may yield an optimal drilling trajectory such as to successfully encounter the desired subsurface feature.

Several topics are investigated further for rigorous direct applications. For instance, R. Ait-Haddou and M.A. Slawinski are elaborating a formalism involving the notions of differential geometry, in particular, following the Finslerian approach via the local Minkowskian properties of this space. Notably, this method yields a geometrically fruitful context for inhomogeneous, anisotropic media by invoking the notion of continuously varying metrics. The initial results are prepared for the PImS PDF workshop in Vancouver (September 1998).

4.6 References

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