Chapter 6

Resistance Monitoring

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6.1 Introduction

MacMillan McGee Corp. is involved in the removal of soil contaminants using a process in which a set of electrodes is placed in the soil and the soil is heated by passing a current through it. The electrical conductivity of the soil, and hence the heat generated, is enhanced by pumping heated water in at some of the electrodes and removing it at a well (or set of wells). The heated water also provides some convected heating, which in the case of soil, far exceeds the conductive heating.

The problem considered was that of estimating the temperature field in the region of interest, the contaminated region, using measurements of electrical potential and current and also of temperature, at accessible points such as the wells and electrodes and the soil surface. These measurements in conjunction with the field equations for the electrical potential and temperature provide a mathematical “Inverse Problem”. There is a known empirical relationship between

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temperature and electrical conductivity so that knowledge of one at any given place and time also determines the other.

On the timescale considered, essentially days, the equation for the electrical potential is static. At any given time the potential V satisfies the equation $\nabla \cdot (\sigma \nabla V) = 0$. Time enters the equation only as a parameter since σ is temperature and hence time dependent.

The problem of finding σ when both the potential V and the current density $\sigma \partial V/\partial n$ are known on the boundary of the domain is a standard inverse problem of long standing. See for example Kohn and Vogelius [4]. It is known, with some restrictions on smoothness, that a unique solution σ exists in spaces of dimension $n \geq 2$. (In one dimension only the harmonic mean of σ can be determined). It is also known that the problem is ill posed and hence that an accurate numerical solution will be difficult especially when the input data is subject to measurement errors. There is much literature on both theoretical and numerical aspects of the problem but we were not able to find a recent comprehensive survey.

In Section 6.5 we examine a possible method for solving the electrical inverse problem which could possibly be used in a time stepping algorithm when the conductivity changes little in each step.

Since we are also able to make temperature measurements there is also the possibility of examining an inverse problem for the temperature equation. There seems to be much less literature on this problem, which in our case is essentially, a first order equation with a heat source. (We neglect thermal conductivity, which is small compared with the convection). Combining the results of both inverse problems might give a more robust method of estimating the temperature in the soil.

Removing soil contaminants using the combined techniques of electrical heating and fluid injection changes the medium in a predictable fashion. On occasion, this method of contaminant removal breaks down and the properties of the medium itself can suffer a sudden change. It is important to be able to detect not only if this situation arises, but also where in the medium the method breaks down.

The equations for the problem are given in the following section.

6.2 Equations

The basic equations of our problem are an energy equation, a quasi-static electrical equation and a porous medium equation. The electrical equation for the potential V is:

$$\nabla \cdot (\sigma \nabla V) = 0 \tag{6.1}$$

where σ is the electrical conductivity. The equations for the fluid flow, D'Arcy's law and incompressibility, are:

$$\vec{u} = -\frac{k}{\mu}\nabla p, \quad \nabla \cdot \vec{u} = 0 \tag{6.2}$$

where $\vec{u}$ is the fluid velocity, p is the pressure, μ is the kinematic viscosity and k is the permeabilty. For the energy balance with Joule heating as a source term we have [2]:

$$\frac{\partial}{\partial t}(\rho_a c_a T) + \vec{u} \cdot \nabla(\rho_w c_w T) = \nabla \cdot (\kappa_a \nabla T) + \sigma \, |\nabla V|^2 \tag{6.3}$$

$\rho_a c_a$	2.5×10^6 J/m^3/°C
$\rho_w c_w$	4.2×10^6 J/m^3/°C
κ_a	1 J/s/m/C
T_0	100°C
T	$10 - 100$°C
V	$100 - 200$ volts
σ	$0.1 - 0.01$ /Ω/m
$\lvert\vec{u}\rvert$	0.3 m/day
ϕ	0.3
S_w	$0.5 - 1$
a	3 /°C

Table 6.1: Some typical numerical values for this problem. Note that 1 watt sec = 1 Joule and that for this problem, a typical length is 10 m. A typical time for the process is 60 days.

where T is the temperature, ρ is the density, c is the specific heat, κ is the thermal conductivity, with the subscripts a and w denoting average and water respectively. The average values are computed as $\rho_a c_a = \phi \rho_w c_w + (1-\phi)\rho_s c_s$, $\kappa_a = \phi \kappa_w + (1-\phi)\kappa_s$ where the subscript s is for soil and ϕ is the volume fraction of water.

In addition to these equations we have an empirical relation between the conductivity σ and the temperature T

$$\sigma = \sigma_w \frac{\phi^{\alpha_n}}{P_\alpha} S_w^2 f(T) + (1-\phi)\sigma_s \tag{6.4}$$

where S_w is the water saturation, σ_s is a constant depending on the soil type, and α_n and P_α are empirically obtained constants. The function $f(T)$ is found to be linearly dominated and is taken to be $f(T) = 1 + a(T - T_0)$. Typically f increases by a factor of three for a 100°C increase in temperature from T_0. Some typical numerical values, supplied by Bruce McGee, are indicated in Table 6.1.

We now scale the energy equation (6.3) by writing

$$\vec{x}' = \vec{x}/L, \quad t' = t/\tau, \quad \vec{u}' = \vec{u}/U, \quad \sigma' = \sigma/\sigma_0, \quad V' = V/V_0, \quad T' = (T - T_0)/\triangle T$$

to obtain the scaled equation (dropping primes):

$$\lambda \frac{\partial}{\partial t} T + \vec{u} \cdot \nabla(T) = \varepsilon \nabla \cdot (\nabla T) + \gamma \sigma \left| \nabla V \right|^2 \tag{6.5}$$

where λ, ε, γ are constants given by

$$\lambda = \frac{\rho_a c_a L}{U \tau \rho_w c_w}, \quad \varepsilon = \frac{\kappa_a}{U L \rho_w c_w}, \quad \gamma = \frac{\sigma_0 V_0^2}{U L \rho_w c_w \triangle T}.$$

If we take $L = 10$ m, $\tau = 20$ days, $U = 0.3$ m/day, $V_0 = 100$ volts, $\sigma_0 = 0.1$ /Ω/m and $\triangle T = 20$°C we find,

$$\lambda \simeq 1, \quad \varepsilon \simeq 0.7 \times 10^{-2}, \quad \gamma \simeq 1/3.$$

Also with these scalings $f(T) = 1 + \alpha T$ with $\alpha = a\triangle T$ and we write $\sigma = \widetilde{\sigma} f(T)$ with $\widetilde{\sigma}$ and α being $O(1)$.

Boundary conditions for the problem are that the potential or current flow is given at all electrodes and at the soil surface. Both being zero at infinity. For the fluid, pressure or velocity is given at input and output wells. For the temperature we assume a known temperature at input wells.

The small value of ε indicates the dominance of convection over conduction in the equations and neglecting conduction gives a first order equation for T. This means that the temperature along the characteristics is governed by either an initial temperature or a fluid temperature at a fluid input. If these temperatures are not compatible, thermal shocks could appear. The small conduction would smooth these shocks in an internal boundary layer of thickness $\varepsilon^{1/2}$. Boundary layers might also occur at output wells but we avoid these by assuming that the temperature at output wells is that of the emerging fluid. In other words we do not apply temperature boundary conditions at output wells.

6.3 A one dimensional example

We consider a one-dimensional problem with two electrodes that coincide with the input and output wells. As boundary conditions we take constant current at the electrodes and a given temperature at the input well. We will assume constant scaled ambient temperature zero as the initial condition. In the one-dimensional case the velocity vector $\vec{u}$ becomes a constant which we take to be 1. We also assume that $\widetilde{\sigma}$ and α are constant.

The potential equation:

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial V}{\partial x}\right) = 0 \tag{6.6}$$

may be integrated to give $I = \sigma \partial V/\partial x = I(t)$ a function of time only. With the given boundary condition I is constant in both space and time. To find V we must first solve for T to obtain σ. V may then be found by another integration.

With I constant, the scaled energy equation (6.5) becomes:

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{I^2}{3\widetilde{\sigma}(1+\alpha T)}. \tag{6.7}$$

This equation may be solved by the method of characteristics. We solve the equation on the domain $t > 0$, $0 < x < L$. The boundary conditions are $T = 0$ on $t = 0$, (ambient physical temperature T_0) $T = g(t)$, $t > 0$ on $x = 0$. The characteristic equations are:

$$\frac{dt}{ds} = 1, \ \frac{dx}{ds} = 1, \ \frac{dT}{ds} = \frac{K}{1+\alpha T} \ \text{ where } \ K = \frac{1}{3}\frac{I^2}{\widetilde{\sigma}}. \tag{6.8}$$

These equations must be solved in two distinct regimes. Fluid already in the soil will be initially at temperature $T = 0$. This fluid moves with unit speed and at any particular time, since all the fluid will have been subjected to the same amount of heating, the temperature will be the same. The appropriate initial conditions for a particle initially at x_0 are, $t = 0$, $T = 0$,

$x = x_0$. Since the input well is at $x = 0$ there will be a front at $x = t$, and this solution will apply for $x > t$. Fluid that has been injected at the input well at time t_i will enter the system with a temperature $T = g(t_i)$, and its temperature will increase as it moves away from the well. This fluid will remain behind the front and boundary conditions are $t = t_i$, $T = g(t_i)$, $x = 0$. The solution is:

$$T(x,t) = \begin{cases} \dfrac{1}{\alpha}(\sqrt{1+2\alpha Kt} - 1), & x > t \\ \dfrac{1}{\alpha}\left[\sqrt{(1+\alpha g(t-x))^2 + 2\alpha Kx} - 1\right], & x < t. \end{cases} \tag{6.9}$$

If $g(0) \neq 0$, T will be discontinuous on $x = t$. A smooth solution may be obtained by including the small conductivity term and introducing an interior layer by means of a substitution $\xi = (t-x)/\varepsilon^{1/2}$. Even when $g(0) = 0$ the derivatives of T will be discontinuous so that a boundary layer will again be required in order to obtain a smooth function T. The change in T, however, will only be of order $\varepsilon^{1/2}$ in this case.

For a specific example we assume fluid is injected at a constant temperature $T_1 > 0$ so that $g(t) = T_1$ for $t > 0$. The potential difference V between the electrodes is given by $V = I\int_0^L 1/\sigma\, dx$. This and the temperature at $x = L$ can be measured. The potential difference is given by:

$$V(L,t) = \begin{cases} \dfrac{L-t}{\sqrt{1+2\alpha Kt}} + \dfrac{1}{\alpha K}\left[\sqrt{(1+\alpha T_1)^2 + 2\alpha Kt} - (1+\alpha T_1)\right], & 0 < t < L \\ \dfrac{1}{\alpha K}\left[\sqrt{(1+\alpha T_1)^2 + 2\alpha KL} - (1+\alpha T_1)\right], & t > L. \end{cases} \tag{6.10}$$

The temperature at $x = L$ is given by:

$$T(L,t) = \begin{cases} \dfrac{1}{\alpha}(\sqrt{1+2\alpha Kt} - 1), & 0 < t < L \\ \dfrac{1}{\alpha}\left[\sqrt{(1+\alpha T_1)^2 + 2\alpha KL} - 1\right], & t > L. \end{cases} \tag{6.11}$$

Measurement of these quantities would allow the calculation of the only unknown quantities α and $\widetilde{\sigma}$. It would also give some indication of whether the assumption that α and $\widetilde{\sigma}$ are constant is a reasonable one.

The quantities above are calculated using $\alpha = 0.4$, $K = 1.25$, $L = 1$ and $T_1 = 0.5$. Since $T_1 \neq 0$ there is a temperature front along the characteristic $x = t$. In Figure 6.1 we show the overall resistance which with constant current is just V. We see that V becomes constant after the front reaches $x = L$. In Figure 6.2 we show the temperature as a function of both time and position. The movement of the front is clear.

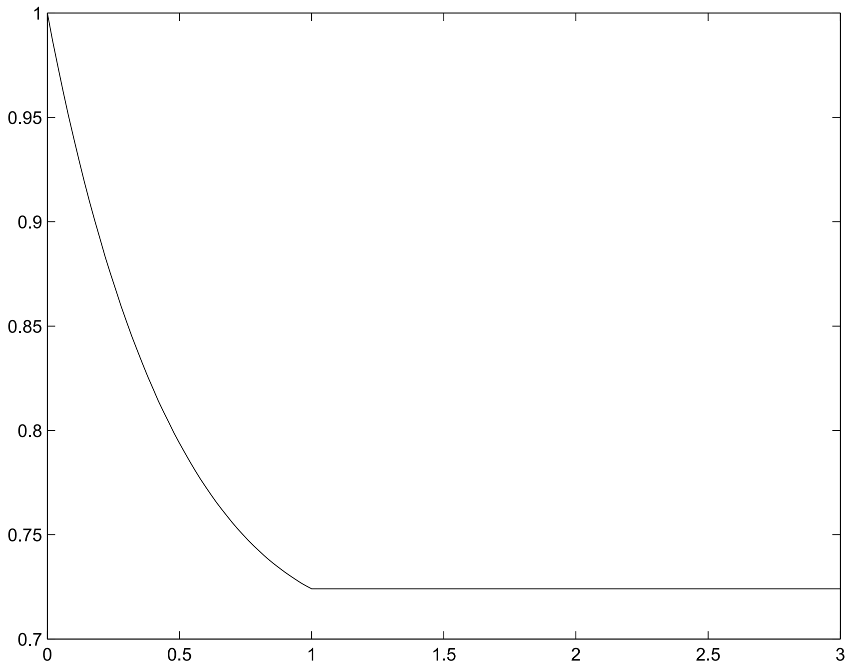

Figure 6.1: Overall resistance.

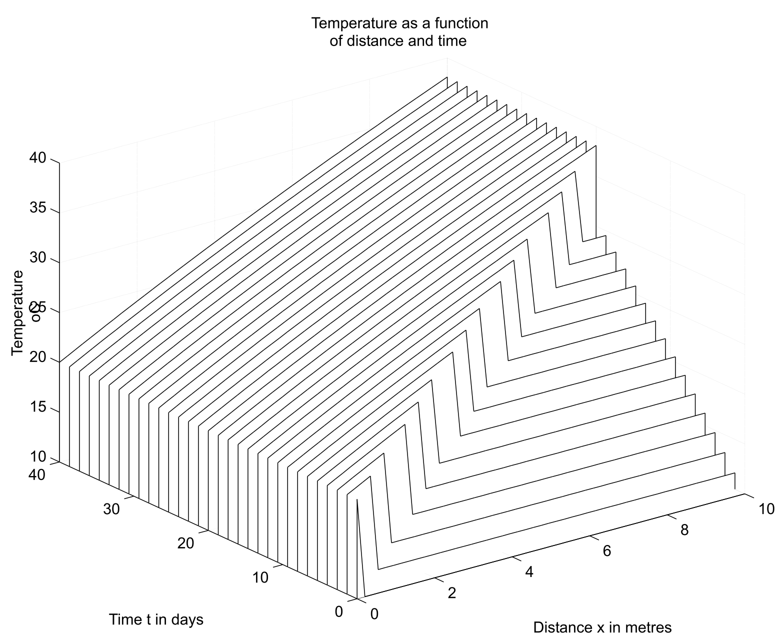

Figure 6.2: Temperature as a function of space and time.

If we prescribe the potenal V at the electrodes the current I becomes a function of time which is determined by integrating the potential equation (6.6) with respect to x from $x = 0$ to $x = L$. If we take the potential difference between the electrodes to be 1, we find

$$I(t) = 1/\int_0^L \frac{1}{1 + \alpha T(x,t)}\,dx. \tag{6.12}$$

The equation (6.7) then becomes non-local and an analytic solution is no longer possible. The equation can however still be solved numerically. With constant $\widetilde{\sigma}$ the results are similar to those for the case of constant current.

In some numerical examples we look at the effects which occur in the current flow and temperature at the well (both measurable quantities) due to an anomaly in the conductivity where we assume that $\widetilde{\sigma}$ has a piecewise change over a region beginning at a specified time. We find that in this simple case of a single anomaly measurements at the boundary are enough to show the time at which the anomaly occurs, its location and its magnitude. With more than one anomaly information may be obtained but there is a loss of uniqueness.

Numerical solutions have been found for an anomaly occurring at time 30 days over the region between 4 and 5 metres. In Figure 6.3 we show graphs of the overall resistance and in Figure 6.4 the output temperature for the case with and without an anomaly which is an increase in scaled conductivity $\widetilde{\sigma}$ from 1.0 to 1.5. Notice that the overall resistance drops at the time the anomaly occurs. The temperatures in the cases with and without anomaly also start to separate at this time. There are also noticeable changes in the temperature at the times that the characteristics through the ends of the anomalous region reach the output well. Since the speed of propagation along the characteristics is known this enables us to locate the anomaly. The percentage change in the resistance at the jump allows us to estimate the percentage change in $\widetilde{\sigma}$.

In Figures 6.5, 6.6 we show similar graphs with an anomaly which is a decrease in scaled conductivity $\widetilde{\sigma}$ from 1.0 to 0.7.

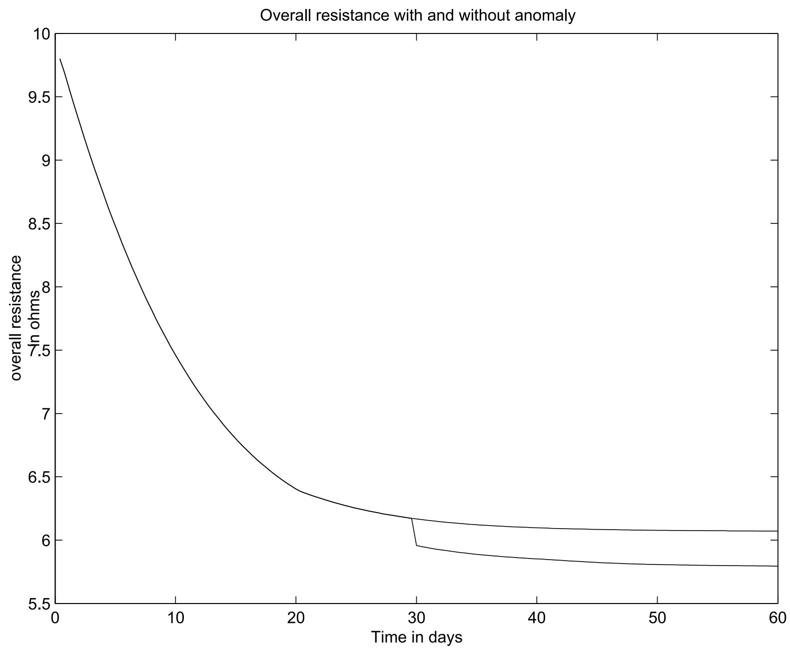

Figure 6.3: Overall resistance anomalous increase in conductivity.

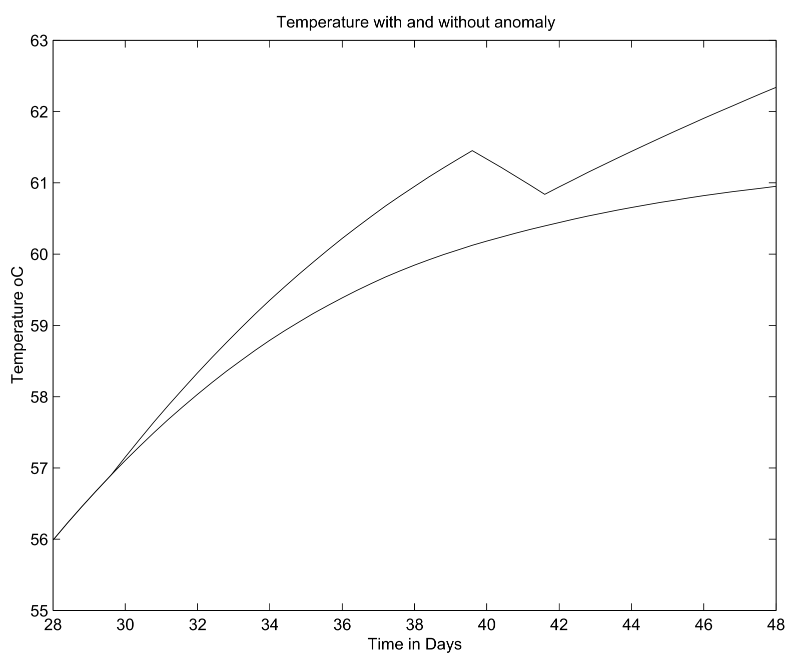

Figure 6.4: Temperature (Zoomed). Anomalous increase in conductivity.

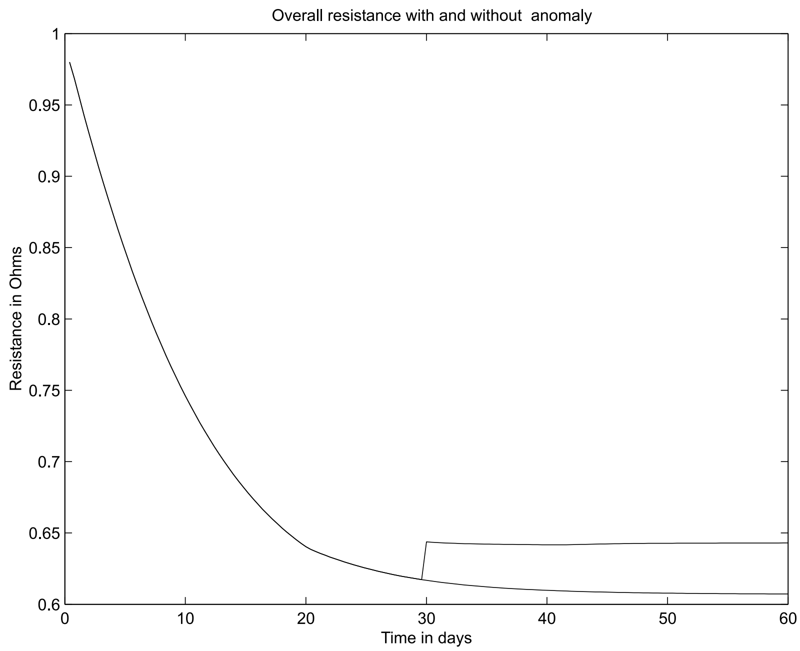

Figure 6.5: Resistance. Anomalous decrease in conductivity.

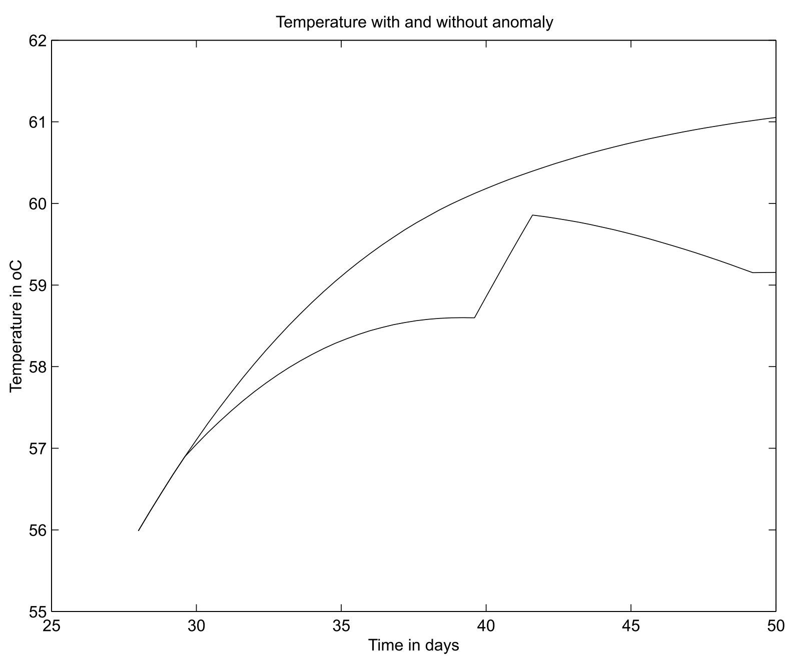

Figure 6.6: Temperature (Zoomed). Anomalous decrease in conductivity.

6.4 The three dimensional case

We make here a few remarks extending the one-dimensional case. These have not been developed but could be the basis for further analysis. The equations for the fluid flow are not coupled to the other equations and the fluid flow is essentially determined by the boundary conditions at the wells. These are Dirichlet, if the pressure head is given, or Neumann, if the velocity is given.

In the case where the permeability, viscosity and electrical conductivity are constant, the pressure head and the electrical potential both satisfy the Laplace equation, if the boundaries and boundary conditions match (i.e. both Dirichlet or both Neumann) the fluid streamlines and the current streamlines will coincide. These streamlines are also characteristic curves for the energy equation and since they will begin and end at a well the energy equation (6.3) may be integrated along a streamline to give:

$$T_2 - T_1 = \int_{r_1}^{r_2} \frac{I^2(r)}{\sigma} \frac{1}{\rho_w c_w u(r)} \, dr \tag{6.13}$$

where r is the distance along a streamline and $u(r)$ is the fluid speed along the streamline. Subscripts 1 and 2 refer to the wells at the beginning and end of the streamline. The temperature T_1 can be controlled, sudden changes in T_2, which should be local due to the small thermal conductivity, would indicate an anomaly along the corresponding streamline (characteristic).

Both the compressibility and potential equation can be integrated along a stream tube to give constant volumetric flow $Q = \int_{\partial A} \vec{u} \cdot \vec{n} \, dA$ and constant total current flow $I = \int_{\partial A} \sigma \nabla V \cdot \vec{n} \, dA$ along a stream tube (here ∂A is a surface perpendicular to the stream tube). These also can be monitored at the wells.

In conditions of steady state temperature one can use the incompressibility equation (6.2) and the potential equation (6.1) to reduce the energy equation (6.3) to:

$$\nabla \cdot \left(\vec{u}T - \frac{1}{\rho_w c_w} \sigma V \nabla V \right) = 0. \tag{6.14}$$

Integrating this along a stream tube we see that

$$\int_{\partial A} \left(\vec{u}T - \frac{1}{\rho_w c_w} \sigma V \nabla V \right) \cdot \vec{n} \, dA = C \tag{6.15}$$

is also constant along a thin stream tube. At a well this is essentially

$$QT - \frac{1}{\rho_w c_w} VI.$$

By monitoring these quantities at the two wells we should be able to detect the occurrence of anomalies in the conductivity. Collating this with the temperature measurements might allow us to pinpoint any anomalies in both time and space.

6.5 The inverse potential problem.

As stated in the introduction, by making a series of impedance measurements about the boundary of the well site an image of the conductivity distribution can be determined. This is complicated by the fact that not all of the boundary is available for measurement. If however we concentrate on sudden changes in conductivity, a detailed image is not required and what follows may be considered rather as a first step in their detection.

6.5.1 Solution of the forward problem and a discussion of the resolving power

To simplify the inverse problem we consider a vertical section of soil of thickness w and cylindrical cross section of radius r_0. The domain is surrounded with N equally spaced electrodes and we assume that a given electrode can either supply current or monitor voltage. Electrodes extend the width of the section and do not overlap ensuring that the width of an electrode, $2a$, satisfies the bound $2aN < 2\pi r_0$. For a given set of electrodes there are many possible applied current patterns. Our assumption will be that a current of I is applied between two of the electrodes and the remainder of the electrodes monitor the resulting voltage at the boundary of the domain.

The forward problem is given a conductivity, $\sigma(r,\theta,z)$, to determine the potential $\varphi(r,\theta,z)$ satisfying

$$\nabla\cdot(\sigma\nabla\varphi)=0 \tag{6.16}$$

and the boundary condition

$$J(\theta)=\sigma\left.\frac{\partial\varphi}{\partial r}\right|_{r=r_0}=\begin{cases}\dfrac{I}{2aw}, & \alpha-\frac{a}{r_0}\le\theta\le\alpha+\frac{a}{r_0}\\ -\dfrac{I}{2aw}, & \beta-\frac{a}{r_0}\le\theta\le\beta+\frac{a}{r_0}\\ 0, & \text{all other } \theta.\end{cases} \tag{6.17}$$

The angles α, β are the angular positions of the electrodes. An estimate of the sensitivity of the potential to changes in the conductivity can be determined by assuming the conductivity is piecewise constant

$$\sigma=\begin{cases}\sigma_0, & r_1\le r\le r_0\\ \sigma_1, & 0\le r<r_1.\end{cases} \tag{6.18}$$

Since both the potential and the current are continuous at $r=r_1$ one has the conditions

$$\varphi|_{r=r_1^-}=\varphi|_{r=r_1^+}, \qquad \sigma_1\left.\frac{\partial\varphi}{\partial r}\right|_{r=r_1^-}=\sigma_0\left.\frac{\partial\varphi}{\partial r}\right|_{r=r_1^+}. \tag{6.19}$$

Solving (6.16) with (6.18)-(6.19) with the constraints that φ is periodic in θ and constant with respect to z yields

$$\varphi(r,\theta)=\varphi_0+\sum_{k=1}^{\infty}f_k(r)[a_k\cos(k\theta)+b_k\sin(k\theta)]$$

where

$$f_k(r)=\begin{cases}\left(1+\dfrac{\sigma_0-\sigma_1}{\sigma_0+\sigma_1}\right)\left(\dfrac{r}{r_1}\right)^k, & 0\le r<r_1\\ \left(\dfrac{r}{r_1}\right)^k+\left(\dfrac{\sigma_0-\sigma_1}{\sigma_0+\sigma_1}\right)\left(\dfrac{r_1}{r}\right)^k, & r_1\le r\le r_0.\end{cases} \tag{6.20}$$

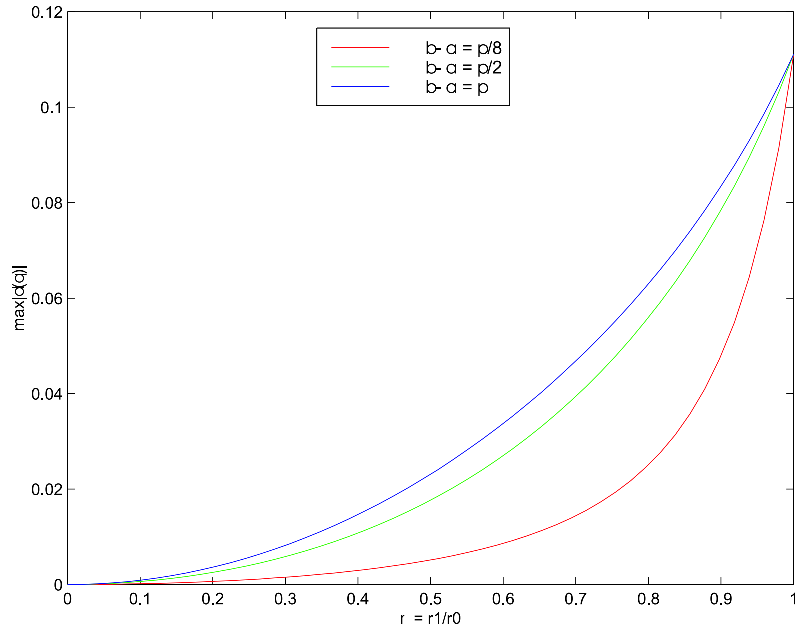

Figure 6.7: Resolving power for various configurations.

Applying the boundary condition (6.17) yields the following expression for the potential on the boundary of the domain

$$\varphi(r_0,\theta) = \varphi_0 + \frac{Ir_0}{\sigma_0 a w}\sum_{k=1}^{\infty}\frac{1+\mu\rho^{2k}}{1-\mu\rho^{2k}}\frac{1}{k^2}\sin\left(\frac{ka}{r_0}\right)\left[\cos k(\theta-\alpha)-\cos k(\theta-\beta)\right]$$

where $\mu = (\sigma_0 - \sigma_1)/(\sigma_0 + \sigma_1)$ and $\rho = r_1/r_0$.

Denote $\overline{\varphi}$ as the potential at $r = r_0$ when $\sigma_1 = \sigma_0$. Using this definition one can define the resolving power as

$$\delta = \frac{\varphi - \overline{\varphi}}{\overline{\varphi}} = \frac{\displaystyle\sum_{k=1}^{\infty}\left[\frac{1+\mu\rho^{2k}}{1-\mu\rho^{2k}} - 1\right]\frac{1}{k^2}\sin\left(\frac{ka}{r_0}\right)\left[\cos k(\theta-\alpha)-\cos k(\theta-\beta)\right]}{\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\left(\frac{ka}{r_0}\right)\left[\cos k(\theta-\alpha)-\cos k(\theta-\beta)\right]}.$$

Figure 6.7 plots $\max|\delta(\theta)|$ as a function of ρ in the case of $N = 16$ electrodes. In addition, $a = 0.1$, $r_0 = 1$ and $\sigma_1/\sigma_0 = 0.9$. Illustrated are cases where the current is applied at adjacent electrodes ($\beta - \alpha = \pi/8$), electrodes at right angles ($\beta - \alpha = \pi/2$) and opposite electrodes ($\beta - \alpha = \pi$). Clearly with an anomaly centered at $r = 0$ current applied at opposite electrodes is the most sensitive.

6.5.2 An iterative technique for solving the inverse problem

The method implemented for solving the inverse problem is to use the sensitivity theorem usually attributed to Geselowitz [3]. For a given region of conductivity σ an application of current I

applied between electrodes A and B generates a potential field $\varphi(\sigma)$ inside the volume. The potential between electrodes C and D is denoted φ_{CD}. By applying the same current between electrodes C and D one obtains the potential ψ_{AB} between electrodes A and B. One may note that

$$\phi_{CD}(\sigma) = \frac{1}{I}\int_{\Omega} \sigma \nabla\, \phi(\sigma) \cdot \nabla\, \psi(\sigma)\, d\Omega = \psi_{AB}(\sigma).$$

If the conductivity of the region changes to $\sigma + \Delta\sigma$ then the change in the transfer impedance between electrodes A, B and C, D is

$$\Delta Z = \phi_{CD}(\sigma + \triangle\sigma) - \phi_{CD}(\sigma) = \phi_{CD}(\sigma + \triangle\sigma) - \psi_{AB}(\sigma) = -\frac{1}{I}\int_{\Omega} \Delta\sigma\ \nabla\psi(\sigma) \cdot \nabla\phi(\sigma + \Delta\sigma)\, d\Omega. \tag{6.21}$$

To solve the inverse problem one assumes a conductivity distribution σ and solves the corresponding forward problem obtaining φ. Taking the difference between the calculated $\varphi(\sigma)$ and the measured $\varphi(\sigma + \triangle\sigma)$ yields ΔZ. In principle we can use this information to solve for $\Delta\sigma$ and find the conductivity distribution.

Consider a mesh $\{\Omega_k\}_{k=1}^{M}$ consisting of M elements each having its own conductivity σ_k. In this way one may approximate the conductivity distribution as a finite sum of characteristic functions

$$\sigma = \sum_{j=1}^{M} \sigma_j \chi(\Omega_j), \qquad \chi(\Omega_j) = \begin{cases} 1, & (x, y) \in \Omega_j \\ 0, & \text{otherwise.} \end{cases}$$

We refer to the collection of conductivities, $\{\sigma_j\}_{j=1}^{M}$, as the vector $\vec{\sigma} \in \mathbb{R}^M$. The corresponding discrete version of the sensitivity theorem (6.21) becomes

$$\Delta Z = -\sum_{j=1}^{M} \Delta\sigma_j \int_{\Omega_j} (\nabla\varphi \cdot \nabla\psi)\, d\Omega_j. \tag{6.22}$$

In this way ΔZ refers to the difference in voltage between electrodes C and D when the input electrodes are A and B and the conductivity changes from σ to $\sigma + \Delta\sigma$.

During the inversion process a number of different current injection patterns are applied. If for example there are E electrodes and current is applied at adjacent electrodes then there are $E-1$ patterns and therefore a total of $(E-1)\times(E-1)$ readings.[1] This collection of measurements on the boundary over the class of chosen patterns generates the vector $\Delta\vec{Z} = \vec{\zeta} - \vec{\varphi} \in \mathbb{R}^N$. Here $\vec{\zeta}$ is the vector of measured voltages and $\vec{\varphi}$ is the vector of the calculated potentials for the current conductivity distribution σ. The final form of the (6.22) is

$$\Delta\vec{Z} = S\Delta\vec{\sigma} \tag{6.23}$$

where the S is an $N \times M$ matrix whose elements are

$$S_{ij} = -\int_{\Omega_j} (\nabla\varphi_i \cdot \nabla\psi_i)\, d\Omega_j \tag{6.24}$$

[1]This assumes the capability of measuring the voltage at the injection electrodes.

the result in cell j obtained when applying pattern i. Current applied to the electrode pair A, B generates φ_i while current applied at C, D generates ψ_i. Because of symmetries $N = E(E-1)/2$. One of the difficulties with the reciprocity theorem is that $S^T S$ is ill conditioned and may be singular if $N \leq M$. Another problem encountered in using the reciprocity theorem is that one cannot use a basic FEM package to solve the forward problem. Direct access to the basis functions of each cell is required to compute S.

Because the problem is ill posed, Tikhonov regularization is used to determine the conductivity. With this method, rather than solving (6.23), one attempts at the kth iteration to minimize the functional

$$f(\Delta\vec{\sigma}_k) = \left\| S\Delta\vec{\sigma}_k - (\vec{\zeta} - \vec{\varphi}_k) \right\|^2 + \lambda \left\| F\vec{\sigma}_k \right\|^2 \tag{6.25}$$

which indicates a trade off with obtaining a solution to $S\Delta\vec{\sigma}_k = \vec{\zeta} - \vec{\varphi}_k$ and not letting $\|F\vec{\sigma}_k\|$ get too large. Typically F is chosen to penalize solutions with properties that one wants to avoid. In our case, we want to avoid solutions with discontinuities so F is a smoothing operator. Since the current implementation is a Laplacian operator, F is an $M \times M$ matrix with $F_{ij} = -1$ if node i is connected to node j $(i \neq j)$ and $F_{ii} =$ number of elements connected to node i. If one lets the regularization parameter $\lambda \to 0$ one obtains a generalized solution of $S\Delta\vec{\sigma}_k = \vec{\zeta} - \vec{\varphi}_k$. We now detail the algorithm used to generate the solution.

Step 1: The first step is to determine an appropriate estimate for a homogeneous conductivity. This is accomplished by setting the conductivity of each cell to one and generating the solutions of the forward problems

$$\nabla \cdot (\nabla\varphi_0) = 0, \qquad \left.\frac{\partial\varphi_0}{\partial\mathbf{n}}\right|_{\partial\Omega} = \vec{J}$$

where each of the applied current patterns, $\vec{J}$, are used in turn. The effective homogeneous conductivity σ_0 is chosen to minimize

$$\left\| \vec{\varphi}_0 \frac{1}{\sigma_0} - \vec{\zeta} \right\|^2. \tag{6.26}$$

Solving (6.26) gives the least squares solution

$$\sigma_0 = \frac{\vec{\varphi}_0 \cdot \vec{\varphi}_0}{\vec{\varphi}_0 \cdot \vec{\zeta}}.$$

Every cell is set to this initial conductivity giving the constant initial vector $\vec{\sigma}_0 = \{\sigma_0\}_{i=1}^{M}$. The iteration parameter $k = 1$.

Step 2: The next step is to generate perturbations in the conductivity. At this point we solve the forward problems

$$\nabla \cdot (\sigma_{k-1}\nabla\varphi_k) = 0, \qquad \sigma_{k-1} \left.\frac{\partial\varphi_k}{\partial r}\right|_{\partial\Omega} = J;$$

for the $E-1$ current patterns and generate the vectors $\vec{\varphi}_k$ by appending the collection the boundary values at the electrodes for each of the applied patterns (as done in Step 1). For

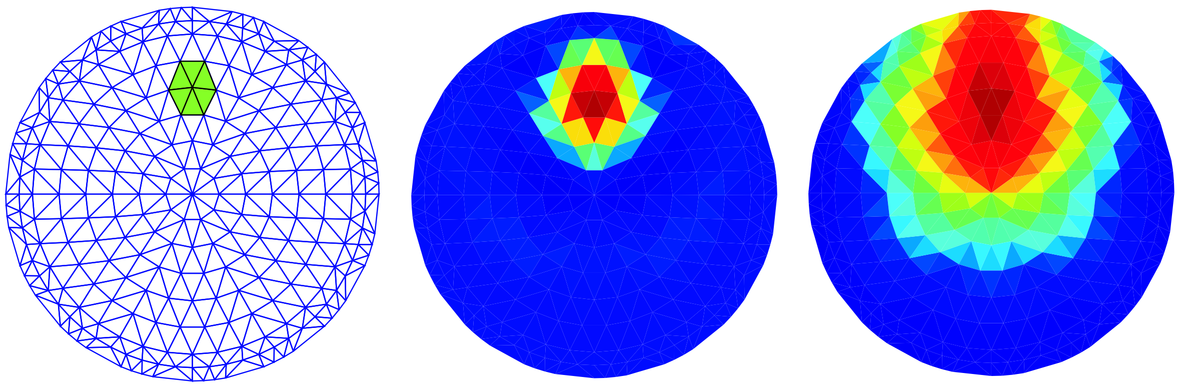

Figure 6.8: Initial anomaly and reconstructed solutions for two values of λ.

all subsequent steps the forward solution in all of the cells is required and not just those corresponding to electrodes. Using the forward solutions one computes the sensitivity matrix, S, from expression (6.24). At this point $\Delta\vec{\sigma}_k$ is chosen to minimize (6.25) and as a result, satisfies

$$\left(S^T S + \lambda F^T F\right) \Delta\vec{\sigma}_k = S^T(\vec{\zeta} - \vec{\varphi}_k) + \lambda F^T F \vec{\sigma}_{k-1}.$$

Step 3: The updated conductivity is set to $\vec{\sigma}_k = \vec{\sigma}_{k-1} + \Delta\vec{\sigma}_k$ and we check the stopping criterion. If $\|\Delta\vec{\sigma}_k\| \geq \epsilon$ then $k \mapsto k+1$ iterate S, $\vec{\sigma}_k$, $\vec{\varphi}_k$ and return to Step 2. Otherwise the loop terminates.

For the numerical simulations we chose $N = 16$ electrodes and a homogeneously conductive region of $\sigma = 1$ with a small conductive anomaly with $\sigma = 0.9$. The mesh has $M = 279$ nodes. Figure 6.8 displays the original anomaly and two reconstructed solutions corresponding to $\lambda = 1 \times 10^{-4}$ and $\lambda = 10$ respectively.

Clearly the cylindrical geometry is an idealization of the actual geometry. However, the promising results of the reconstructed solutions shown here indicate that detecting sudden large-scale changes in conductivity is certainly feasible. Moreover, for a given configuration of electrodes, estimates can be obtained for the size of the smallest detectable anomaly of a given strength.

6.6 Summary

Effort was focussed on the detection of sudden localised anomalies in the conductivity and progress was made on a simple one dimensional model for the coupled problem and on a two dimensional model of the uncoupled potential problem. These investigations showed some promise and could be developed further.

Since substantial literature exists on the potential problem, further work should begin with a thorough literature search. The MacMillan McGee problem may present some novel aspects. Since measurements will only be made on an actual decontamination process, multiple current inputs will not be available. Measurements will also only be available at electrodes, wells and the soil surface so that the effect of boundary conditions at "infinity" should be considered. This all adds to the illposedness (see Cheney, Issacson and Newell [1])

The use of temperature as well as potential measurements for the inverse problem might also prove fruitful. This would require an efficient algorithm for the numerical solution of the three dimensional coupled forward problem.

Changes in overall resistance similar to that found in the one dimensional example have been observed in practice and it would be interesting to measure temperatures at the output wells to see whether effects predicted by the one dimensional case do actually occur and whether they are spatially localised.

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