Mathematical Models of Mother/Child Attachment

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1 Introduction

Attachment theory is a branch of psychology in which the bond between one person with another is studied. Of particular importance is the attachment of a child with her mother.[2] The form of this attachment will not only affect the ability of the mother to regulate the anxiety of the child, but, also, it has been postulated that the child uses her mother as a secure base from which she can explore her world [1, 2, 9]. Furthermore, the attachment that a child forms with her mother during the first year of her life will not only affect a child's relationship with her mother, but will affect the attachments she forms with others for the rest of her life.

As part of a larger observational study, Ainsworth developed the "strange situation" laboratory experiment to probe the attachment of a child with her mother [1]. In the experiment, a toddler's response to her mother and the environment is observed during a series of high and low anxiety situations. Of particular interest, is how the child interacts with her mother to help regulate her anxiety, and the quality of the child's play and exploration of her new environment.

The strange situation takes place over a period of twenty minutes, and consists of a series of eight episodes: (1) the mother and child enter a laboratory playroom that contains

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[2]Although, for consistency, we will refer to the child as a female, the theory does not distinguish the sex of the child, and the experiments described in this report involved children of both sexes. Thus, the discussion throughout this report is meant to hold equally regardless of the sex of the child.

toys, (2) a stranger enters the room, and attempts to engage the child, (3) the mother leaves the room for approximately 3 minutes, during which time the stranger attempts to comfort/engage the child if the child appears distressed, (4) the mother re-enters and attempts to comfort the child, (5) the child and mother stay for a time in the room, while the child is once again free to play, (6) after a time, the mother and the stranger leave the room, (7) while the mother is still not present the stranger re-enters the room and attempts to comfort the child, (8) the mother re-enters the room and attempts to comfort the child.

Based on the observations of the experiment, the children are placed into three categorizes of attachment, (1) secure, (2) avoidant, or (3) ambivalent. The avoidant and ambivalent categories correspond to insecure attachment. Upon first entering the laboratory playroom, a child exhibiting secure attachment will be willing to explore the room, coming back to her mother or glancing at her mother periodically. When the mother returns after her short absence, such a child will seek out her mother, and will subsequently calm down rapidly. Once calm, the child will quickly return to play and exploration of the room. A child exhibiting avoidant attachment will tend not to seek contact with her mother during the first phase when she first enters the room with her mother. However, the quality of play and exploration is not as high as that of the secure child. During her mother's absence, the avoidant child may not show any outward signs of distress, and when the mother re-enters the room, she may not show signs that she has noticed her return. Indeed she may avoid eye contact with her mother. However, the avoidant child will still seek out her mother if she feels a heightened sense of anxiety for a prolonged period of time. Upon initially entering the room, an ambivalent child will tend to hover close to her mother, and will only be willing to explore the new environment very tentatively. Although such a child will seek out the mother upon her return after the short absence, she will not immediately derive comfort from her mother, and may even show outward signs of hostility, such as hitting her mother, or pushing her mother away.

The heart of attachment theory is that the category of attachment in which a child falls is correlated with the history of the mother's response to the child's requests for comfort during times of increased anxiety. As the main part of her study, Ainsworth made home-based observations, noting the quality of a mother's interaction with her child. She found that if a mother consistently responded to her child's call for attention with sensitivity, the child tended to develop a secure attachment. If a mother consistently tended to ignore the child's calls for comfort, the child tended to develop an avoidant attachment. If a mother sometimes responded to her child with sensitivity, and sometimes did not, the child tended to develop an ambivalent attachment.

The presenters of this problem requested the academic participants to develop mathematical models of the decision-making and behaviour that occur when a child is subjected to a stressful episode, e.g. as in the strange situation. The overall goal was to determine the factors that are most relevant in determining the type of attachment the child exhibits toward her mother, and to determine whether these factors are associated with the sensitivity and consistency with which the mother generally responds to the child's requests for comfort and attention. Such a model would shed light on the mechanism by which the mother's behaviour influences the character of her child's attachment to her, and it would provide evidence in support of attachment theory itself.

This report is structured as follows. In Section 2, we use game theory to probe the mother and child's decision making process in the hopes that we can classify the three distinct mother-child behaviours. In Section 3, we develop a dynamical system that models the relevant features that determine the child's response to situations of varying levels of stress, e.g. as experienced in the "strange situation." The goal is to reproduce the distinct dynamics associated with the three different attachment categories as parameters related to the mother's sensitivity and consistency are varied. Finally, in Section 4, we give a brief introduction to the application of control theory to this problem.

2 Game theory

In this section, we apply game theory to the decision making process of the child and mother. The analysis of payoffs and strategies associated with game theory and the later theory of moves has been used for decades in many fields such as economics, diplomacy, religion, politics and biology (see [3, 6, 15, 16, 18] for overviews and discussion). There are two key ideas that arise from such analyses. The first is that of an equilibrium strategy where both players (in this case for a two-player game) have no incentive to change their strategy and receive a different payoff. The second idea is that of moving or changing strategies. Of the two, the second requires additional assumptions about the rationality, motivations and desires within each player, and some of these can be contested. Needless to say, we will focus on the former and dodge the latter. We will use game theoretic approaches to try and understand how behaviours like "secure", "ambivalent", and "avoidant" emerge as equilibrium choices for the child, given her upbringing. We focus on the situation immediately after the mother has returned to the room. At this moment, the child must decide whether or not to approach her mother to seek comfort. At the same time, the mother must decide whether or not to attend to the child.

2.1 A one-person game. First, we describe the simplest possible game. Here, the child has to choose what to do (what "strategy" to use) given the situation she's been presented with. This corresponds to the clinical observation that the mother's behaviour is independent of the child's. To start, we define the game. This requires modelling the situation, modelling the child's strategies, and constructing a payoff matrix for the child. Strictly speaking, this scenario is an optimization problem for the child rather than a game theoretic problem because only the child is playing; however, we shall use the game-theoretic language throughout this section.

When her mother re-enters the room, the child is anxious because she'd been left alone with a stranger for two minutes. The mother has two possible strategies: to attend to her child or to ignore her. The mother executes these strategies with probability $\vec{\mathbf{q}} = (q, 1-q)$ where $0 \leq q \leq 1$. If $\vec{\mathbf{q}} = (1, 0)$ then the mother is the Perfect Mother: unfailingly attentive. If $\vec{\mathbf{q}} = (0, 1)$ then the mother is made of stone. Real mothers would have $0 < q < 1$. Whether or not the mother attends in a particular instantiation of the game would be determined by the flip of a q-weighted coin.

In a one-person game, the player (the child) knows what her opponent (the mother) will do and needs to choose a response. Given this information, the child has to choose a strategy (choose an action). For each possible strategy the payoff is the amount of comfort she would receive — the amount that her anxiety would be reduced — should she choose that strategy. In using a one-person game as a model, we are bearing in mind that the mother and child come to the experiment with a long history of prior stressful situations. The child already has a measure of her mother — she knows her mother's $\vec{\mathbf{q}}$.

2.1.1 *A child with two choices.* We first consider a model in which the child has only two choices: to go to her mother seeking comfort or to not to go. The child's payoff matrix

		Mother's action	
		Attend	Ignore
		q	$1-q$
Child's action	Go	1	$-s$
	Don't Go	0	0

Table 1 The child's payoff matrix for the one-person game in which she has two choices.

is given in Table 1. We have normalized the payoffs by the amount of comfort or stress reduction the child receives when she seeks comfort and her mother attends to her. Thus, it is possible for the value in the upper left to be -1 in the pathological case where the mother comforting the child *raises* the child's stress level. When the child seeks comfort but her mother ignores her, the payoff is $-s$. If the child is stressed by such rejection, then $s > 0$. If she would be comforted by being close to her mother, even if she is being ignored, then $s < 0$. We do assume that $s > -1$: being ignored by her mother provides less comfort than if her mother attends to her. Finally, if the child does not go to her mother for comfort then she receives no comfort, whether or not her mother attends to her.

The model has two parameters: q and s. These reflect the parenting strategy and the quality of the mother-child interaction, respectively.

Since the model is a one-person game, the optimal strategy will be a pure strategy: the child will always go to her mother or will never go to her mother. This is to be contrasted with a "mixed" strategy in which the child would go to her mother with probability p where $0 < p < 1$.

If $s \leq 0$ then "Go" is a dominant strategy: no matter what the mother does, the child is better off seeking comfort than not. If $s > 0$ then there is no dominant strategy. What the child chooses to do will depend on the values of q and s.

Assume $s > 0$. The child's expected payoffs are

$$\begin{aligned} P_{\text{Go}} &= q - (1-q)s, \\ P_{\text{Don't Go}} &= 0. \end{aligned}$$

This means that if the child always plays "Go" then after many plays of the game, the average payoff per game will be approximately P_{Go} and the more times the game has been played, the closer the average payoff will be to P_{Go}. We see $P_{\text{Go}} > P_{\text{Don't Go}}$ if and only if $s < q/(1-q) =: s_{\text{crit}}$.

At first sight, it seems odd to imagine that the child plays this game many times (which is required in order to use P_{Go} and $P_{\text{Don't Go}}$ to analyze the child's options.) Indeed, if one only counts the number of times that the child goes through the "Strange Situation" protocol then it would be unusual. We suggest, however, that the mother and child played this game many times before they reached the laboratory situation — every time the child cried out for food, a fresh diaper, reassurance, or entertainment. The "Strange Situation" protocol is designed to try and extract the essence of these interactions, rather than being a completely novel situation. And so we view it as yet another play of a game played many times before. Secondly, it may seem odd to imagine that a child is capable of statistical thinking. It is our understanding that children as young as six months old have demonstrated statistical inference in early language acquisition.

If the child seeks comfort but is ignored this causes her stress s. If this stress is less than s_{crit} then she is better off seeking comfort and risking stress than not — her strategy is always "Go". If $s > s_{\text{crit}}$ then her optimal strategy is always "Don't Go". If $s = s_{\text{crit}}$ then it doesn't matter what she does; on average she will have zero stress reduction whether she chooses "Go" or "Don't Go".

It remains to understand how the parenting strategy, q, determines s_{crit}. If the mother is unfailingly attentive ($q = 1$) then the child should always seek comfort because $P_{\text{Go}} > P_{\text{Don't Go}}$ no matter what the value of s. If the mother is never attentive ($q = 0$) then the child should never seek comfort because $P_{\text{Go}} < P_{\text{Don't Go}}$ (recall that we've assumed $s > 0$). If the mother is generally quite attentive then q is close to 1 and s_{crit} is large. This means that the child can have a relatively large stress response to rejection (s) but still be better off always seeking comfort because the chances of rejection are relatively low. Similarly, if the mother is generally inattentive then q is close to 0 and s_{crit} is small. This means that only the thick-skinned child would be better off choosing "Go" — only a child with a low stress response to rejection can afford the risk.

If q and s are such that $s > s_{\text{crit}}$ then we would say the child is "avoidant" and observe that there is no stress mediation because the child's expected payoff is 0. If $s < s_{\text{crit}}$ then the child is either "secure" or "ambivalent". In both of these behaviours, the child goes to her mother for stress reduction.

We finish our discussion by noting that in practise one would not expect the child's stress response, s, to be independent of the mother's parenting strategy q. This, in turn, makes it hard to predict what the child's behaviour will be. If the mother is reliably attentive then s_{crit} would be large and one might expect the child to be secure. But one might also expect the child of such a reliably attentive mother to be truly stressed out should her mother not attend to her: her s could be quite large and might exceed s_{crit}. Similarly, if the mother is not at all reliable in her attentiveness then s_{crit} would be close to zero. But one might also expect the child of such a mother to have become quite thick-skinned: her s could be quite small. We also note that the child's stress response will also have a biological component, one that is not caused by her upbringing. Different people have different pain thresholds and sensitivities; this would certainly enter into the child's value of s. This biological component would provide a lower bound on the child's s value. As a result, it provides a threshold for how unreliable the child's mother could be: if q becomes too close to zero then s_{crit} will be less than the biological component of the child's stress response and the child will be avoidant no matter what.

2.1.2 *A child with three choices.* We now consider what might happen if the child had a third strategy, one of guarded behaviour.

From the video footage, the ambivalent child is observed to approach her mother for comfort, and yet is not entirely willing or able to accept full comfort. For example, we see the child asking to be picked up, but holding a hand between herself and her mother, as if making sure that her mother does not get too close. At mildly stressful times, the child stays close to the mother but does not interact with her. We call this the "Half-Go" action: the child is making a guarded request for comfort. If the child chooses the "Half-Go" action and her mother attends to her then she gets some reduction of stress, but not as large a reduction had she approached in an unguarded manner. On the other hand, if her mother ignores her, then she is not as upset as she would have been had she approached in an unguarded manner. And so, the payoffs for the "Half-Go" strategy would be somewhere between the payoffs for the "Go" and "Don't Go" strategies.

The child's payoff matrix is shown in Table 2. The payoffs for the "Go" and "Don't Go" strategies are as before (Section 2.1.1). If the child half-goes to the mother for comfort

		Mother's action	
		Attend	Ignore
		q	$1-q$
Child's action	Go	1	$-s$
	Don't Go	0	0
	Half-Go	h	$-t$

Table 2 The child's payoff matrix for the one-person game in which she has three choices.

and her mother attends to her, then her payoff is h where $0 < h < 1$. If she half-goes to her mother and is ignored, then her payoff is $-t$. As before, there are two cases for the sign of t. If $t < 0$ then she receives comfort from being around her mother even if her mother ignores her; in this case, we assume $-1 < s < t < 0$. If $t > 0$ then she is stressed by her mother's ignoring her; in this case, we assume $0 < t < s$.

Is it ever rational for the child to choose this new strategy? As before, if $t \leq 0$ then "Go" is a dominant strategy. No matter what the mother does, the child should never choose "Half-Go" or "Don't Go".

We now consider the $0 < t < s$ case, for which there is no dominant strategy. If the child plays the game many times, always choosing the same option, then her expected payoff will be one of

$$\begin{aligned} P_{\text{Go}} &= q - (1-q)s, \\ P_{\text{Don't Go}} &= 0. \\ P_{\text{Half-Go}} &= hq - (1-q)t. \end{aligned}$$

Figure 1 gives the graphs of these expected payoffs as a function of q. In both plots, the

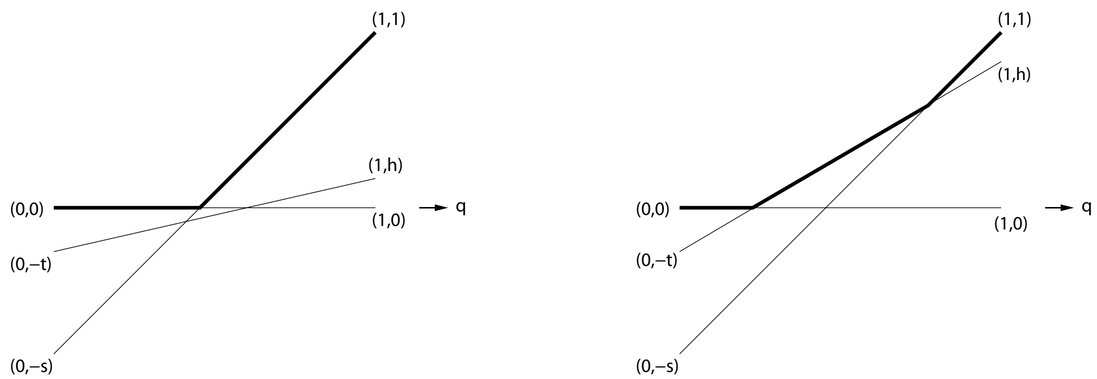

Figure 1 The lighter lines are the graphs of the expected payoffs P_{Go}, $P_{\text{Half-Go}}$, and $P_{\text{Don't Go}}$ for the one-person game with the payoff matrix given in Table 2. The heavy curve is the maximum expected payoff. Left: $h < t/s$, Right: $h > t/s$.

line connecting $(0, -s)$ to $(1, 1)$ is the graph of P_{Go}, the line connecting $(0, -t)$ to $(1, h)$ is $P_{\text{Half-Go}}$, and the line connecting $(0, 0)$ to $(1, 0)$ is $P_{\text{Don't Go}}$. The heavy curve is the graph of the maximum expected payoff — this curve gives the child's optimal strategy for each value of q. Either there are two optimal behaviours possible, as in the left plot of Figure 1, or there are three, as in the right. To distinguish between these situations, one finds that the graphs of P_{Go} and $P_{\text{Half-Go}}$ intersect at (q_c, P_c) where $q_c = (s-t)/(1-h+s-t)$ and $P_c = (sh-t)/(1-h+s-t)$. If $h < t/s$ then $P_c < 0$ and there are only two possible optimal strategies: whatever the value of q, $P_{\text{Half-Go}}$ is always less than either P_{Go} or $P_{\text{Don't Go}}$. If $h > t/s$ then $P_c > 0$ and there is an interval of q values for which the "Half-Go" strategy is optimal. The balance between h and t/s reflects the balance between the decrease of comfort received when a guarded request is attended to $(h/1)$ and the decrease in stress caused when a guarded request is ignored (t/s).

Here is a way to view the condition for viability of the "Half-Go" strategy. Consider a mixed strategy of "Go" and "Don't Go" for the child, which we shall call "Random-Go", designed to offer the same average level of comfort to the child as does "Half-Go" *in those cases when the mother attends*. That is, the child chooses "Go" with probability h and chooses "Don't Go" with probability $1-h$. When the mother attends, the payoff to the child is $1 \cdot h + 0 \cdot (1-h) = h$, which is the same as for "Half-Go". How does "Random-Go" compare overall with "Half-Go"? When the mother ignores, the child's payoff under "Random-Go" is $-s \cdot h + 0 \cdot (1-h)$, which equals $-sh$. On the one hand, if $-sh$ is better than the payoff under "Half-Go", $-t$, then it can be seen that "Random-Go" dominates "Half-Go". That is, if $-sh > -t$ (equivalently, if $h < t/s$), then the child will never want to use the "Half-Go" strategy. On the other hand, if $h > t/s$, then "Half-Go" dominates "Random-Go"; indeed, it turns out that for some values of q, "Half-Go" is better than any mixed strategy that combines "Go" and "Don't Go". If "Random-Go" had been designed to offer the same average level of comfort to the child as does "Half-Go" in those cases when the mother ignores, then one would find that the child would choose "Go" with probability t/s and one would come to the same conclusion about $h < t/s$ and $h > t/s$.

In Section 2.1.1, we analyzed the game in terms of the child's stress response, s, relative to a critical stress level, $s_{\text{crit}} = q/(1-q)$. In doing this, we had taken the parenting strategy, q, as fixed. We also took as fixed the child's stress reduction when she sought attention and her mother attended to her. This was a little less obvious because it was reflected in the 1 in the payoff matrix of Table 1. In fact, this was originally a free parameter that became 1 when it was used to normalize the entire payoff matrix.

If the child has three options, there are five parameters: q (the parenting strategy), 1 and h (the stress reductions when attended to), and s and t (the increases in stress when requests for comfort are ignored). In Figure 2 we hold q, 1, and h fixed and find that the threshold behaviours for s and t are determined by s_{crit} and hs_{crit}. In the plot to the left, we have taken $h = 1/3$, but we get the same qualitative picture as long as $0 < h < 1$. In this case, the region $0 < t < s$ is divided into three regions, with one strategy being optimal in each region. To better understand the situation, we consider the extremal cases of $h = 0$ and $h = 1$. As h decreases to 0, the points A and B converge to the point $(s_{\text{crit}}, 0)$ and "Half-Go" is no longer an optimal strategy. The resulting $h = 0$ plot is shown in the top right of Figure 2. The reason for the disappearance of "Half-Go" is apparent if one looks at the left plot of Figure 1: as $h \to 0$, the point $(1, h)$ moves down to $(1, 0)$ and $P_{\text{Half-Go}}$ will always be less than $P_{\text{Don't Go}}$, whatever the values of s and t. (This can also be seen directly from the payoff matrix.) At the other extreme, as h increases to 1, the points A and B converge to the points $(0, 0)$ and $(s_{\text{crit}}, s_{\text{crit}})$ respectively and "Go" is no longer an optimal strategy. The resulting $h = 1$ plot is shown in the bottom right of Figure 2. One can either understand this by taking $h \to 1$ in the right plot of Figure 1 or by noting that if $h = 1$ in the payoff matrix in Table 2 then the "Half-Go" strategy dominates the "Go" strategy, whatever the values of s and t (with $s > t$).

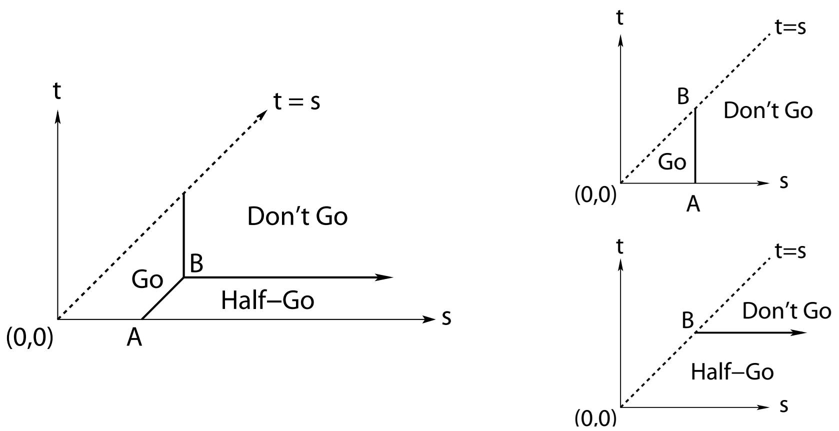

Figure 2 In all three plots, the dashed line reflects the bound $t < s$. The points A and B are $((1-h)s_{\text{crit}}, 0)$ and $(s_{\text{crit}}, hs_{\text{crit}})$ respectively. Left: $h = 1/3$. There are three regions, with one strategy being optimal in each region. Right: There are only two possible optimal strategies. Top: $h = 0$. Bottom: $h = 1$.

2.2 A Two-Person Game. In using our one-person game as a model, we took the mother's parenting strategy as fixed and known to the child. We now take into account that attending to the child is at a cost to the mother and that she might choose her strategy in a way that reflects both this cost and what she thinks her child might do. To do this, we use a two-person game with two payoff matrices, one for the child and one for the mother. It should be noted that notions like "optimality" may not be appropriate in multiplayer games where dominating strategies can cycle (see [18] for a broad overview and discussion). For instance, strategy A may dominate B meaning that A will always yield a payoff greater than strategy B. But, C may dominate A and B may dominate C. As we shall see, some situations in the game model yield dominant strategies for both players, and some will not.

As in Section 2.1, the mother executes her options, "Attend" and "Ignore", with the strategy $\vec{\mathbf{q}} = (q, 1-q)$ where $0 \le q \le 1$. If $q = 0$ or 1 then $\vec{\mathbf{q}}$ is called a "pure" strategy and otherwise is called "mixed". Similarly, the child executes her options, "Go" and "Don't Go", with the strategy $\vec{\mathbf{p}} = (p, 1-p)$. Whether or not the child seeks comfort from her mother in a particular instantiation of the game is determined by the flip of a p-weighted coin.

			Mother's action	
			Attend	Ignore
			q	$1-q$
Child's action	Go	p	$(1, 1-c)$	$(-s, -s)$
	Don't Go	$1-p$	$(0, -c)$	$(0, 0)$

Table 3 The payoff matrices for the two-person nonzero-sum game. For each combination of choices, the payoff is represented as an ordered pair. The child's payoff is the left element of the ordered pair and the mother's payoff is the right element. For instance, if the child goes to the mother and the mother attends, the child's payoff is 1 and the mother's is $1-c$.

Table 3 gives the payoff matrices for the child and her mother. The child's payoff matrix is the same as in Section 2.1.1. The mother's payoff matrix reflects two things. First, whatever increase or decrease there is in the child's stress levels is also a gain or a loss for the mother, hence the 1 and the $-s$ in the mother's payoff matrix. In other words, we assume the mother accurately interprets the child's responses to attention and develops her strategy accordingly. In addition, attending to her child takes energy and takes time away from some other activity she might prefer doing, hence the $-c$ in the payoff matrix.

To analyze the game, we use concepts from the non-cooperative theory of nonzero-sum games. The mother-child game is considered non-cooperative because the players are unable to make a binding agreement on a joint choice of strategy. Using Table 3, we can construct the individual payoff matrices for the players. Each entry of the matrix is an ordered pair. The child's payoff is the first element of each ordered pair, and the mother's payoff is the second element. We denote the child's payoff matrix by A and the mother's payoff matrix by B, so

$$A = \begin{bmatrix} 1 & -s \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1-c & -s \\ -c & 0 \end{bmatrix}. \tag{2.3}$$

If the players play the strategies $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ then the child's payoff is $\vec{\mathbf{p}}^{\mathrm{T}} A \vec{\mathbf{q}}$ and the mother's payoff $\vec{\mathbf{p}}^{\mathrm{T}} B \vec{\mathbf{q}}$. This is not a zero-sum game where "money" into one player's pocket comes out of the other player's pocket: $A \neq -B$. Our discussion will focus on the payoffs for Nash equilibria, which are pairs of strategies where a unilateral change in strategy will not improve the player's payoff *even if the player knows the opponent's strategy.* One interesting difference between zero-sum games and nonzero-sum games is that in a zero-sum game, if there are two Nash equilibria $(\vec{\mathbf{p}}, \vec{\mathbf{q}})$ and $(\vec{\mathbf{p}}^*, \vec{\mathbf{q}}^*)$ then their payoffs are the same: $\vec{\mathbf{p}}^{\mathrm{T}} A \vec{\mathbf{q}} = \vec{\mathbf{p}}^{*\mathrm{T}} A \vec{\mathbf{q}}^*$. We will see below that because the mother-child game is nonzero-sum, one can have two pairs of Nash equilibria which have different payoffs.

To simplify the game further, we can reduce the payoffs into broad categories where we replace the values of the payoff with an integer from best (4) to worst (1) denoting the relative size of the payoff to either the mother or the child. For instance, if $-s < 0$, the ordinal representation for the child's payoff is always $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$. The same information could be presented as $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ because decisions are based on the relative payoffs; the relative order of the elements is the critical feature. We classify all possible outcomes using a binary tree representation where each branch corresponds to a comparison between entries. In the case where s is negative, we assume that $-s < 1$ because should the child approach her mother, it would only make sense for the payoff to be greater if the mother pays attention than if she did not. Type I and II is distinguished by whether $0 < -s < 1$, branching left, and $0 > -s$, branching right. Similarly, subtypes A and B are distinguished by $1 - c < -s$, branching left and $1 - c > -s$, branching right. Sub-types 1 and 2 are distinguished by $1 - c < 0$ and $1 - c > 0$, branching left and right, respectively. Finally, sub-sub-types "a" and "b" are distinguished by $-c < -s$ and $-c > -s$, branching left and right, respectively. The full classification is shown in Figure 3. In some cases, the ordinal representation is the same for all sub-subtypes. In these cases, "Type I B" refers to all subordinate "Type I B 1 a," "Type I B 1 b," etc...

We show all eight payoff configurations in Figure 4 grouped into four broad categories by the positions of the Nash equilibria.

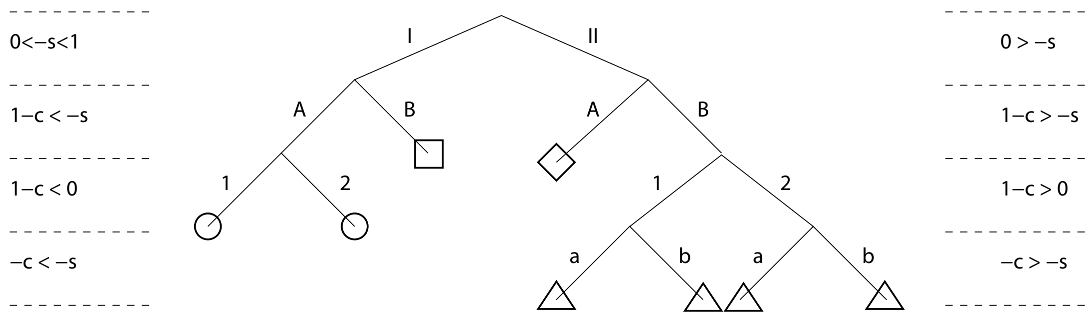

Figure 3 A diagram of possible parameter configurations for the payoff matrix given in Table 3. Every terminal node is a distinct configuration. Terminal nodes are coloured by groups listed in Figure 4: Group 1 (circles), Group 2 (square), Group 3 (diamond) and Group 4 (triangle).

Group 1:

(4,2)	**(3,4)**
(2,1)	(2,3)

Type I A 1

(4,3)	**(3,4)**
(2,1)	(2,2)

Type I A 2

Group 2:

(4,4)	(3,3)
(2,1)	(2,2)

Type I B

Group 3:

(4,2)	(2,3)
(3,1)	**(3,4)**

Type II A

Group 4:

(4,3)	(2,2)
(3,1)	**(3,4)**

Type II B 1 a

(4,3)	(2,1)
(3,2)	**(3,4)**

Type II B 1 b

(4,4)	(2,2)
(3,1)	**(3,3)**

Type II B 2 a

(4,4)	(2,1)
(3,2)	**(3,3)**

Type II B 2 b

Figure 4 The payoff matrices in Table 3 can be reduced to four general categories. Nash equilibria are indicated in bold.

For all Type I scenarios where $0 \leq -s \leq 1$, the first row of the child's payoff matrix is greater than the second row. Thus, the "Go" strategy dominates the "Don't Go" strategy and the child should always play "Go". The mother should choose whichever option is better for her, "Ignore" or "Attend". In the first grouping in Figure 4, it is better for her to "Ignore" and in the second group it is better for her to "Attend". In the third group, the mother's "Ignore" strategy is dominant for Type II A scenarios. Given this situation, the child is always better off with "Don't Go". The remainder of our discussion focuses on the Type II B configurations. This famous game is often referred to as The Battle of the Sexes (see [3] for example, for a discussion of many interpretations). In fact, Bernard applies this game to marriage counselling in [4] in a context very similar to attachment.

A pair of strategies, $(\vec{\mathbf{p}}^*, \vec{\mathbf{q}}^*)$, is called a Nash equilibrium if neither player can obtain a higher payoff by unilaterally changing her strategy — if she knew her opponent's strategy and her opponent could not change her strategy, would she then change her own strategy? Nash equilibria occur when,

$$\begin{aligned} \vec{\mathbf{p}}^{*\mathrm{T}} A \vec{\mathbf{q}}^* &\geq \vec{\mathbf{p}}^{\mathrm{T}} A \vec{\mathbf{q}}^* && \text{for all probability vectors } \vec{\mathbf{p}}, \\ \vec{\mathbf{p}}^{*\mathrm{T}} B \vec{\mathbf{q}}^* &\geq \vec{\mathbf{p}}^{*\mathrm{T}} B \vec{\mathbf{q}} && \text{for all probability vectors } \vec{\mathbf{q}}. \end{aligned}$$

Because neither player would switch her strategy unilaterally, the players can move away from a Nash equilibrium only through some form of cooperation or intervention.

It is easy to find all Nash equilibria that are pure strategies. If $A_{i_0 j_0}$ is greater than or equal to all other entries in the j_0 column of A and if $B_{i_0 j_0}$ is greater than or equal to all other entries in the i_0 row of B then the pure strategies $(\vec{\mathbf{p}}^*, \vec{\mathbf{q}}^*)$ with $p_{i_0} = 1$ and $q_{j_0} = 1$ are Nash equilibria. Doing this, we find that our two-person game has two Nash equilibria. First, there is $p = 0$ and $q = 0$ ("Don't Go" and "Ignore") where the payoff to each player is 0. Second, there is $p = 1$ and $q = 1$ ("Go" and "Attend") where the payoff to the child is 1 and the payoff to the mother is $1 - c$.

There is another Nash equilibrium that is easy to find: $(\vec{\mathbf{p}}^*, \vec{\mathbf{q}}^*)$ where both $\vec{\mathbf{p}}^*$ and $\vec{\mathbf{q}}^*$ are "equalizing" strategies. An equalizing strategy is one that gives the same expected payoff to your opponent, no matter what your opponent does. (Note that you need to know your opponent's payoff matrix in order to find your equalizing strategy.) The child seeks a strategy $\vec{\mathbf{p}}^*$ such that

$$(\vec{\mathbf{p}}^{*\mathrm{T}} B)_1 = (\vec{\mathbf{p}}^{*\mathrm{T}} B)_2 \Longrightarrow p^* - c = -sp^*,$$

and the mother seeks a strategy $\vec{\mathbf{q}}^*$ such that

$$(A\vec{\mathbf{q}}^*)_1 = (A\vec{\mathbf{q}}^*)_2 \Longrightarrow (1+s)q^* - s = 0.$$

The child's equalizing strategy is $p = c/(1+s)$ and the mother's is $q = s/(1+s)$. (This Nash equilibrium only applies to Group 4 scenarios because p and q are probabilities having a range of $[0, 1]$.) The child's expected payoff is 0 and the mother's expected payoff is $-cs/(1+s)$. If $c > 0$ or $s > 0$ this Nash equilibrium is a pair of mixed strategies and is therefore distinct from the two Nash equilibria corresponding to pure strategies.

It is not hard to check that these three are the only Nash equilibria in this game, but dominating strategies, strategies maximizing the payoffs for the individual players, vary depending upon parameter values.

Now, we can summarize the behaviours that would result from the four types of games in the two-player model shown in Figure 4.

Group 1: There is only one Nash equilibrium, the pure strategy pair "Go" - "Ignore". This would appear as a secure relationship because the mother regulates the child's stress, even though the mother is ignoring the child.

Group 2: There is only one Nash equilibrium, the pure strategy pair "Go" - "Attend". This would appear as a secure relationship as well.

Group 3: There is only one Nash equilibrium, the pure strategy "Don't Go" - "Ignore". This is an avoidant relationship.

Group 4: There are three Nash equilibria, two pure pairs "Go" - "Attend" and "Don't go" - "Ignore" as well as the mixed, equalizing strategy. The first two correspond to secure and avoidant relationships just as in the scenarios in Groups 1-3. Unlike the other scenarios, one of the players receives the best possible payoff and the other player receives a payoff that is less than the best possible payoff. This could lead to resentment if the players are non-myopic, meaning the mother and child are capable of understanding the outcomes of other combinations of decisions. The mixed, equalizing strategy has the qualities of an ambivalent relationship. The probability that the child seeks comfort is proportional to c so the more it costs the mother to attend, the clingier the child will be. Likewise, the probability that the mother attends to the child is proportional to s. The more stress the child experiences by rejection, the more likely the mother is to provide comfort.

We have identified equilibrium configurations in the two-player model for mother-child interactions. We have not gone further to hypothesize how mother and child move from one strategy to another because such speculations would go well beyond anything we could defend for two reasons. One, we have no psychological data on how the subjects formulate or adjust their strategies. Two, the mathematical theory of determining equilibria in game theory is quite simple when compared to what is known about formulating strategies based on these equilibria. For instance, on page 359, McKinsey points to a payoff matrix qualitatively equivalent to our Group 4 scenarios, and points out that "It must be remarked that Nash's theory ... has serious inadequacies and certainly cannot be regarded as a definitive solution of the conceptual problems in this domain... The theory of Nash seems to throw little light on the question of how to play a game having such a pair of payoff matrices" [16]. Certainly, one can argue that pure strategies explain the existence of secure and avoidant relationships, and that once these strategies are employed, it would be irrational for either the mother or child to change unilaterally in certain parameter regimes in Groups 1-3 regimes. The dynamics underlying situations like Group 4 are the subject of continued investigation and deliberation. For instance, Steven Brams offers one model for negotiated solutions in ordinal games called the Theory of Moves to find non-myopic equilibria that avoid cycling or to determine a unique resolution for both players [6, 7, 8, 13]. In our model, the mixed strategy has features resembling an ambivalent relationship. While the process by which the strategies of the mother and child evolve over time remains fertile ground for investigation, we see that this simple two-player, non-cooperative game with two simple parameters yields a rich mathematical structure that recovers three principal attachment behaviours observed in clinical experiments.

3 A dynamical systems approach

The "strange situation" experiment was designed to help examine a child's relationship with her mother under some stressful conditions. In particular, the experiment can be used to examine the mother's ability to down-regulate the anxiety of her child. The source of the anxiety in the experiment is due to the mother leaving the room. The child's reaction to the mother's return to the room is very telling of the child's type of attachment to her mother, and how much comfort and security she derives from her mother. In particular, children's response to this situation correlates well to their general modes of behaviour, described in Section 1.

To attempt to mathematically model and distinguish these behaviours we must consider some measurable quantity available from the data of the experiment. One such quantity is the physical distance between the mother and the child. However, physical distance is not necessarily a sufficient indicator of the child's ability or need to derive comfort from her mother, because, for instance, visual contact may serve the same purpose. Thus, we consider instead "emotional distance". This concept along with a deterministic model that gives the time evolution of this quantity, will be described in detail in Section 3.2. Another measurable quantity of interest is the child's level of anxiety. In the following section, we consider a simplified model that temporarily ignores the distance to focus on the child's anxiety level as a function of time, $A(t)$.

3.1 The anxiety equation. In this section we derive an equation giving the evolution of the anxiety A. Details of the definition of anxiety are left open as some aspects may be specific to different situations, but generally an increase in anxiety will imply a decrease in comfort and feelings of well-being. The anxiety level may be measured using some externally visible measure of the child's comfort or emotional state, although this may be difficult, or, perhaps more quantitatively, via physiological variables such as heart rate or the level of cortisol in the child's saliva. The base-line of anxiety may be determined by considering the child in a familiar situation, say at home with the mother.

As a first version for the equation describing the evolution of anxiety, we write

$$m\frac{d^2A}{dt^2} + b\frac{dA}{dt} + \frac{1}{r}(A - \bar{A}) = S(t), \qquad A(t = 0) = \bar{A}, \qquad A'(0) > 0. \tag{3.1}$$

Here $S(t) \geq 0$ is a function describing the externally imposed stress on the child; in particular we take it to be an indicator function of the times when the mother is absent, see Figure 5(left). The initial anxiety is assumed to be the baseline level with a tendency for increasing anxiety (positive slope) due to entering an unfamiliar environment.

The parameter m is a measure of the maturity or emotional stability of the child. The parameters r and b determine how the child recovers to her baseline emotional state after a temporary stressful situation. The parameter b is related to the insensitivity of the mother to the child's needs, while the parameter r corresponds to a notion of mother-child distance. The parameter r is in fact related to the ability of the mother to down-regulate the child's anxiety, and is likely to change depending on the immediate situation, and thus, in Section 3.2, we will consider it as a variable and derive an equation to model its time evolution. These parameters and their interpretation are discussed further in the following sections.

This model is analogous to a mechanical system describing a damped-driven harmonic oscillator, a classic problem considered in basic control theory [11]. In such a system, m is the mass, b is a damping coefficient, $1/r$ is a spring constant for a linear restoring force and $S(t)$ is an external driving force. The motivation for this model is that for some parameter values, the anxiety following the stress-event can be sustained (with slow decay) near the peak induced anxiety, while for other values, the anxiety can rapidly decay. This is in agreement with the observed behaviours for insecure (ambivalent and avoidant) and secure infants respectively.

The fact that (3.1) is a linear equation is also convenient because an analytic solution is possible:

$$A(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + A_p(t), \tag{3.2}$$

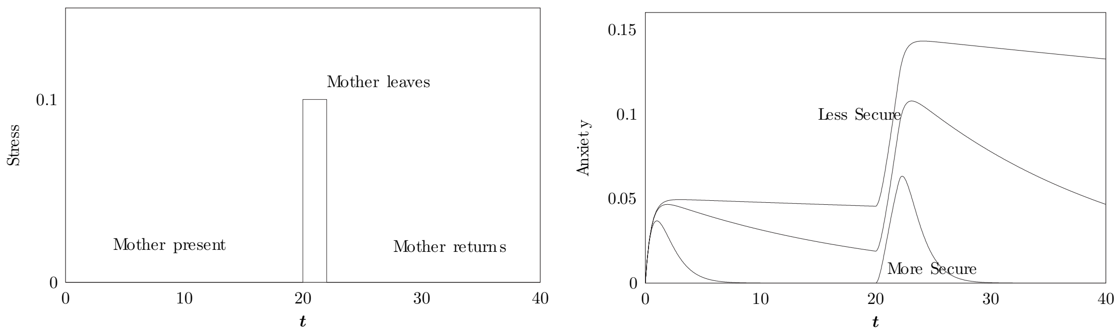

Figure 5 (Left) Stress generated by the temporary absence (3 minutes) of the mother, (Right) Anxiety response generated by (3.1) with m, b fixed for different values of r (the rate of anxiety decay is inversely related to r). An increase in r corresponds to an increasingly less secure anxiety response.

where c_1 and c_2 are constants that depend on the initial conditions, the eigenvalues λ_i are given by

$$\lambda_{1,2} = \frac{-b}{2m}\left[1 \pm \sqrt{1 - \frac{4m}{rb^2}}\right], \tag{3.3}$$

and $A_p(t) = r\bar{S} + \bar{A}$ is a particular solution that must be included when an imposed stress is present (i.e. when the equation is inhomogeneous; see, e.g., [5]). A mathematical artifact of (3.1) that is inappropriate for a model of anxiety is that oscillatory solutions are possible if the parameters are in the "under-damped" regime [5]. To exclude this possibility, we must be in the "over-damped" regime, and thus we require that the parameters satisfy

$$rb^2 > 4m. \tag{3.4}$$

That is, our model is valid when m is small, i.e. when the child has little ability to self-regulate her anxiety, which is expected to valid for children of the ages that are considered.

Given the assumption (3.4), if $b, r > 0$, unforced solutions will exhibit only exponentially decaying modes, dominated by the slowest decaying mode:

$$\lambda_2 = -\frac{b}{2m}\left[1 - \sqrt{1 - \frac{4m}{rb^2}}\right] \sim \begin{cases} -1/br & \text{if } rb^2 \gg 4m \\ -b/(2m) & \text{if } rb^2 \sim 4m \end{cases} \tag{3.5}$$

In Figure 5(right), we present a simulation of (3.1), for a stress profile, given on the left hand side of the figure, that mimics that of the strange situation. That is, the induced stress is zero before the mother leaves the room, jumps to a high constant value while the mother is absent, and again drops to zero when the mother returns. The different anxiety profiles are produced by varying the parameter r while the other parameters are held fixed. As r is increased, it can be seen that (1) the steady state of anxiety increases, and (2) that the rate of decay of anxiety decreases, in particular, the rate decay is inversely related to r.

We conclude this section with a nonlinear generalization of (3.1),

$$\frac{d}{dt}\left(m\frac{dA}{dt}\right) + \left[\rho^2 A^2 + b\right]\frac{dA}{dt} + \frac{1}{r}\left[(A - \bar{A})_+\right]^\beta = S(t). \tag{3.6}$$

The changes in this model allow for possible effects not considered in (3.1):

- The damping term (i.e., the second term on the left hand side) has been generalized to be of van der Pol type for $\rho > 0$. In this case the damping will increase with increasing anxiety, and thus the larger the child's anxiety becomes the more difficulty the child will have in recuperating.
- Finally, the restoring force (i.e. the third term on the left hand side) has been generalized to describe a nonlinear stress-anxiety (emotional strain) response parametrized by the exponent β. Three regimes are

$$\begin{cases} 0 < \beta < 1 & \text{“Hard case”: a child with a higher tolerance for stress} \\ \beta = 1 & \text{“Linear”: direct proportional response to stress} \\ \beta > 1 & \text{“Soft case”: a child more likely to develop high anxiety} \end{cases} \tag{3.7}$$

  A series of simulations indicating these three cases is shown in Figure 6. In order for these cases to hold, we require $A < 1$, i.e., that the anxiety variable A is scaled by its maximum value. Also note the use of the "one-sided spring model"

$$(A - \bar{A})_+ = \begin{cases} A - \bar{A} & \text{if } A > \bar{A} \\ 0 & \text{else} \end{cases} \tag{3.8}$$

  This describes the situation that if a child's anxiety somehow falls below $\bar{A}$, then the restoring force would not work to drive it up (back to $\bar{A}$).

A simulation of (3.6) with $r > 0$ and $\rho = 0$ is shown in Figure 7. Here the imposed stress $S(t)$ describes the mother leaving and returning twice, first at $t = 20$, then again at $t = 40$. Parameter values are selected to yield "secure" behaviour for $\beta = 1$; anxiety rises when the room is entered ($t = 0$) and then the mother leaves, but in all cases it rapidly decays. For comparison, a "soft" child, with $\beta > 1$ but other parameters unchanged, exhibits a much slower decay of anxiety with some accumulation being evident, as in the case of insecure children. Conversely, a "hardened" child ($\beta < 1$) exhibits more rapid dissipation of anxiety than the linear case. Even more strikingly, their anxiety can go *below* their baseline level; for capturing this behaviour and avoiding spurious oscillations the use of (3.8) is essential.

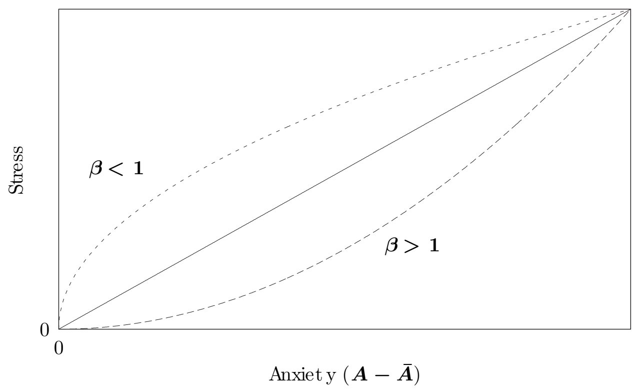

Figure 6 β-dependence of the stress-anxiety response.

At a phenomenological level, these second order differential equations seem to give a reasonable description of the anxiety response to imposed stresses in the experiment. As the parameters of the model described in this section are varied, the solutions describing

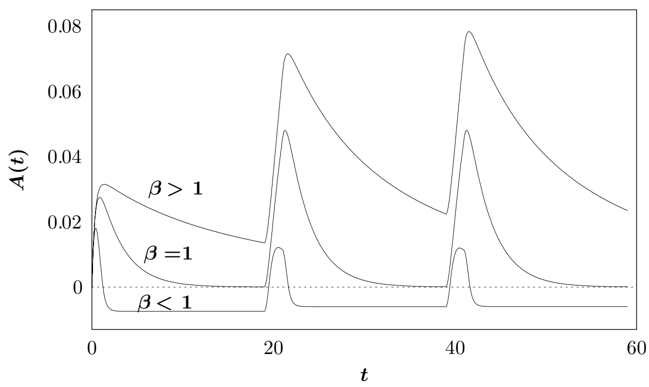

Figure 7 The influence of changing β in (3.6) for a linearly-secure child subjected to successive stresses.

the response to the imposed stress is similar to the responses of a secure or insecure child. However, the model cannot distinguish between ambivalent and avoidant children as their anxiety response should be somewhat similar. In this section, the parameters are assumed to be constant; this is a convenient simplification, but in general, they may exhibit both gradual evolution on long timescales as well as dynamics on shorter time intervals. In particular, there is expected to be short-term variation of r. It is this that we model in the next section. The variety of responses of r to the remaining parameters will distinguish between all attachment types.

The essential element in describing the nature of attachment is relating these parameters of the child's behaviour to properties of the mother's parenting behaviours. This will be the focus of Sections 3.2 and 3.3.

3.2 The mother-child distance equation. The previous section describes an equation for the time evolution of a child's anxiety in a mother-child relationship. The coefficient of the "force" in (3.1) and its nonlinear extension (3.6) is written in terms of a "distance" variable r. This notion of distance is closely related to the actual physical distance between the mother and the child. However, we introduce instead a concept of "emotional distance" between the mother and the child. This concept is more flexible and includes long range interactions between a mother and her child. For instance, visual and verbal contacts between the mother and the child can be effective means by which a mother can aid in the regulation of her child's anxiety, even when the child is not physically close to the mother.

We derive an equation for the child's distance to the mother in terms of the mother's parenting style. The main parenting parameters to be used are

- a =inconsistency of the mother's response to the child's need, and
- b =maternal insensitivity,

where the maternal insensitivity parameter b has already been used in the damping coefficient of (3.1). The emotional distance is described by the variable $r = r(t)$. We will assume that the equation for the time evolution of emotional distance is given by the differential equation

$$\frac{dr}{dt} = \frac{b}{r} - aAr + \frac{c}{1 + A - \bar{A}}, \tag{3.9}$$

where $A = A(t)$ is anxiety, $\bar{A}$ is again the baseline anxiety of the child and c is the intrinsic curiosity of the child at baseline anxiety.

The first term on the right hand side of (3.9) describes the attraction/repulsion the child feels depending on the parenting style of the mother. For high b values (high maternal insensitivity), this term represents a strong impetus to increase emotional distance r when r is not large. However, the effect of this term decreases rapidly as r increases, indicating that a child will still look to her mother to help her regulate her anxiety, in particular in situations of high anxiety. For b close to zero, the effect of this term only becomes important for values of r close to zero. We take $b > 0$, which ensures that if $r > 0$ initially, then it will be so for all time. A mother will have a high value of b if she does not respond to her child's demands during times of anxiety, e.g. the mother is dismissive, responds grudgingly, or ignores the child. A yet higher value of b will result if a mother's response to the child's demands leads to increased anxiety, e.g. a mother becomes angry at the child for demanding attention, and shouts at the child, or in extreme cases, physically harms the child. A low value of b is associated with a parenting style in which the demands of the child are met with sensitivity from the mother.

The second term on the right hand side of (3.9) describes the decrease of the child's emotional distance r as a factor of both anxiety A and the current emotional distance. Thus, we assume there is a natural tendency for the emotional distance to decay exponentially with rate constant aA. The larger the anxiety, the parameter a, or the emotional distance, the faster the decay will be. The parameter a depends on the parenting style of the mother. This parameter will be larger if the mother's response to the child's demands during a time of anxiety is inconsistent, i.e. sometimes the mother responds positively, and sometimes the mother responds grudgingly, dismissively, or ignores the child. In this situation, the emotional distance will tend to decrease relatively quickly even in situations of moderate anxiety, i.e. the child will tend to seek anxiety regulation even when anxiety is not high, because of the uncertainty associated with acquiring the regulation. In the case when a parent provides either consistently positive or consistently negative response, the value of the parameter a will be lower, and the emotional distance will decrease less rapidly for a given value of A. This reflects that for a given level of anxiety, the child will be less likely to seek comfort, either because she knows she will obtain comfort when she needs it, or because she knows that she will not obtain comfort when she seeks it. However, regardless of the value of a, i.e. for all children, this term will represent a large impetus to reduce emotional distance r when anxiety A is high.

Finally, the last term on the right hand side of (3.9) describes the child's impetus to explore. This term will be small when the anxiety is large, i.e. the child will tend to explore only in low anxiety situations.

In Figure 8, we plot the right hand side of (3.9) as a function of r, for various values of the parameters a and b, with anxiety A fixed at its baseline $\bar{A}$. The values of r at which the graph crosses the horizontal axis correspond to values for which $dr/dt = 0$, and thus represent equilibrium solutions of (3.9). Figure 8 shows the smallest such solution; a second equilibrium exists for a larger value of r. However, we do not try to find an interpretation for this second solution at this stage. We see that children with comparatively low a and b values and children with similar low a value but with a higher b value reach close equilibria. However, if both a and b are raised, the curve crosses the r axis at a significantly lower value. To further elucidate the effect of the parameters on the equilibrium solution we set $dr/dt = 0$ in (3.9), and solve for r to obtain the equilibrium surface $r = \phi(a, b)$. See

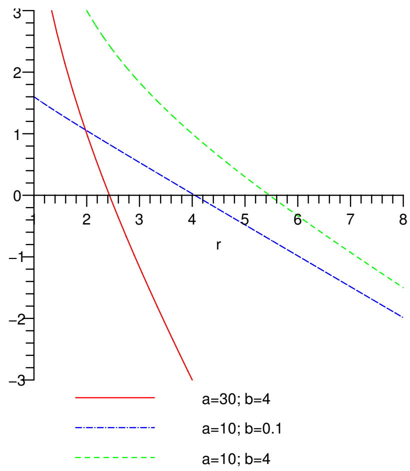

Figure 8 The right hand side of (3.9) is plotted as a function of r, for various values of a and b with A held fixed at baseline ($A = \bar{A} = 0.05$).

Figure 9. It is clear from Figure 9 that the location of equilibrium solutions for (8) depend

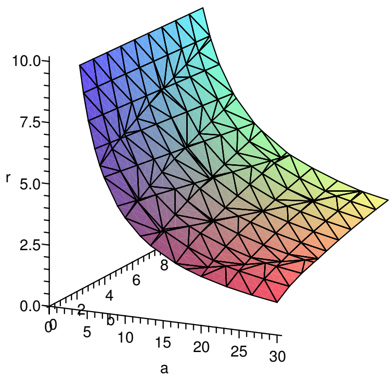

Figure 9 Equilibrium solutions of (3.9) plotted as a function of a and b.

almost exclusively on the parameter a. That is, the child's preferred emotional distance at a time of low anxiety is determined to a large extent by the inconsistency of the mother, while the maternal insensitivity has relatively little effect. Thus, the child of an inconsistent mother (high a) will gravitate to a low emotional distance even at a time of low anxiety.

3.3 The Complete Model. We now turn to the analysis of the full model incorporating anxiety and emotional distance. In the case in which we consider the linear anxiety equation (3.1), we have

$$m\frac{d^2A}{dt^2} + b\frac{dA}{dt} + \frac{1}{r}(A - \bar{A}) = P\tanh(r) + S(t),$$
$$\frac{dr}{dt} = \frac{b}{r} - aAr + \frac{c}{1 + A - \bar{A}}, \tag{3.10}$$

where we have added a term $P\tanh(r)$ in the anxiety equation which corresponds to the increase of anxiety that is induced as the emotional distance is increased, where $P > 0$ is a constant. The function $\tanh(r)$ is used to model the saturation of imposed stress on the child as the emotional distance r becomes large.

Figure 10 shows numerical simulations of system (3.10) for values of a and b that distinguish between the three types of mother-child attachment: secure, ambivalent, and avoidant. The simulations begin at a time of high stress, e.g., in the strange situation experiment when the mother re-enters the room after being absence for a short time. Unlike in Section 3.1, we assume that the absence of the mother induces the same level of anxiety in all the children, regardless of their type of attachment to their mother. Thus, we take the initial conditions to be the same in all simulations. As is seen in Section 3.1, the effect of the mother's return on the anxiety A distinguishes the secure child from the ambivalent and the avoidant child; the secure child's return to baseline anxiety is very rapid, while the anxiety of an avoidant or ambivalent child shows slower decay. However, the anxiety response is not sufficient to distinguish the avoidant child from the ambivalent child. From the graph of emotional distance r, we see that the dash-dot curve represents an initial decay of r that is much more rapid than that of the dashed curve, and the minimum value of r for the dash-dot curve is significantly smaller than that of the dashed curve. We conclude that the dash-dot curve describes the dynamics of the anxiety A and emotional distance r of an ambivalent child, because we expect such a child to seek out her mother, even though she is only moderately comforted by her. We also conclude that the dashed curves exhibit the dynamics of an avoidant child, because we expect that, in a situation of high anxiety, such a child would not wish to be too far from her mother, although she would still wish to maintain a 'buffer zone' between her and her mother, and we would expect her anxiety to be dissipated relatively slowly. Thus, the secure child is characterized by a mother with a parenting style with low inconsistency a and low insensitivity b, an ambivalent child is characterized by both a higher inconsistency a and higher insensitivity b, while the avoidant child has low inconsistency a, but a higher value of insensitivity b.

3.4 Analysis of the complete model. We now proceed to a linear stability analysis of the equilibrium solution of the complete model (3.10).

3.4.1 *Existence and uniqueness of equilibrium solution.* We suppose that $S(t) \equiv 0$ and find the equilibrium solutions of system (3.10) rewritten as a first-order system:

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= \frac{1}{m}\left(-bB - \frac{1}{r}(A - \bar{A}) + P\tanh(r) + S(t)\right) \\ \frac{dr}{dt} &= \frac{b}{r} - aAr + \frac{c}{1 + A - \bar{A}}. \end{aligned} \tag{3.11}$$

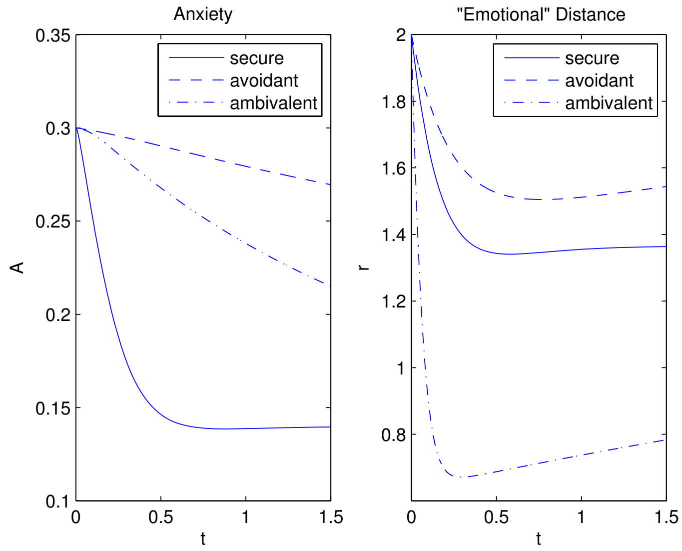

Figure 10 Numerical simulations of the full system as the mother returns to the room. Parameter values are fixed at $c = 2$, $m = 0.001$, $\bar{A} = 0.05$ and $P = 0.075$. Then a and b are varied; 'secure': $a = 10$, $b = 0.1$, 'avoidant': $a = 10$, $b = 4$, 'ambivalent': $a = 30$, $b = 4$.

Set $B = 0$, then the second equation yields

$$A = rP\tanh(r) + \bar{A}.$$

Substituting in the last equation we have

$$G(r) = \frac{b}{r} - a\left(rP\tanh(r) + \bar{A}\right)r + \frac{c}{1 + rP\tanh(r)} = 0. \tag{3.12}$$

For $b, r > 0$ and all other parameters nonnegative, we have that $\lim_{r\to 0} G(r) > 0$ and

$$\frac{dG}{dr} = -\frac{b}{r^2} - a(P\tanh(r) + rP(1 - \tanh(r)^2))r - a(rP\tanh(r) + \bar{A}) - \frac{c(P\tanh(r) + rP(1 - \tanh(r)^2))}{(1 + rP\tanh(r))^2}$$

is negative. This guarantees the existence of a unique equilibrium solution for all values of the parameters.

Figure 11 shows $G(r)$ of (3.12) plotted as a function of r for various values of a and b, where, as before, we set the parameter values $c = 2$, $m = 0.001$, $\bar{A} = 0.05$ and $P = 0.075$. The points at which the graph crosses the r-axis represent solutions of (3.12), and thus, equilibrium solutions of the full system (3.10). The dotted curve, for which $a = 30$ and $b = 4$, intersects at $r \approx 1.27$, the solid curve, for which $a = 10$ and $b = 0.1$, crosses at $r \approx 1.36$ and the dashed curve, for which $a = 10$ and $b = 4$, crosses at $r \approx 1.96$. Let $r = r^*$ be the solution to (3.12) and $A^* = r^*P\tanh(r^*) + \bar{A}$. Then, the equilibrium solution is at $(A, B, r) = (A^*, 0, r^*)$.

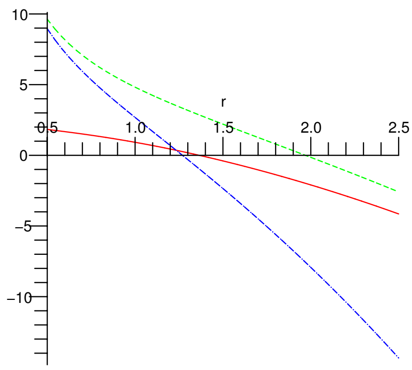

Figure 11 Plot of $G(r)$ indicating the equilibrium solutions of (3.12); dash-dot: $a = 30$, $b = 4$, solid: $a = 10$, $b = 0.1$, and dash $a = 10$, $b = 4$.

3.4.2 *Linear stability analysis.* We now compute the linear stability at this equilibrium solution. The linearization at $(A^*, 0, r^*)$ is

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{1}{r^* m} & -\dfrac{b}{m} & \dfrac{A^* - \bar{A}}{m(r^*)^2} + P\tanh'(r^*) \\ -ar^* - \dfrac{c}{(1 + A^* - \bar{A})^2} & 0 & -\dfrac{b}{(r^*)^2} - aA^* \end{pmatrix}. \tag{3.13}$$

Setting $m = 0.001$, $c = 2$, $\bar{A} = 0.05$ and $P = 0.075$, the characteristic equation for L is

$$p(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

where $a_0 = 1$, and

$$\begin{aligned} a_1 &= \frac{aAr^2 + b + 100r^2 b}{r^2} \\ a_2 &= \frac{100(baAr^2 + r + b^2)}{r^2} \\ a_3 &= \frac{-1}{r^3(A + 0.5)^2}\big(25b + 100bA + bA^2 + 75rA + 150aA^2r^2 + 375\tanh(r)r^3 \\ &\quad +125ar^4\tanh(r) + 5ar^4A^2\tanh(r) - 12.5\,ar^2 + 200aA^3r^2 - 37.5\,r \\ &\quad +50ar^4A\tanh(r)\big)\,. \end{aligned}$$

If $b, r > 0$, one can easily verify that $a_1, a_2 > 0$. The Routh-Hurwitz criterion [12] states that all the eigenvalues of L have negative real parts if and only if

$$a_0 > 0, \quad \Delta_1 = a_1 > 0, \quad \text{and} \quad \Delta_2 = a_1a_2 - a_3 > 0.$$

We verify asymptotic stability for the parameter values of the numerical simulations of Figure 10; the equilibrium solutions have coordinates $(A, 0, r) \approx (0.13, 0, 1.27)$ for $a = 30$,

$b = 4$, $(A, 0, r) \approx (0.14, 0, 1.36)$ for $a = 10$, $b = 0.1$ and $(A, 0, r) \approx (0.19, 0, 1.96)$ for $a = 10$, $b = 4$. Using Maple, it is easy to compute that in all cases Δ_2 is a positive number of the order 10^5. We see that the equilibrium solution is linearly stable in all these cases. Therefore, after a perturbation (within the stability basin) the dynamics relax back to the equilibrium solution.

These stable equilibrium solutions indicate the long-time behaviour of the model (3.10). That is, they indicate the anxiety A and emotional distance r that we expect the child to have after they recover from the stressful event. If we correlate the different parameter values with the different attachment types as we did for the simulations displayed in Figure 10, we see that, in this case, the ambivalent child has the lowest emotional distance and anxiety, while those of the avoidant child are the highest. This is indeed what we might expect; the increased equilibrium anxiety of the secure child relative to that of the ambivalent child is due to the extra anxiety induced by 'exploration', i.e. the increased emotional distance. It should be noted that the equilibrium for the secure child depends significantly on the exploration parameter c, while the equilibrium for the ambivalent child is less sensitive to this parameter. Specifically, an increase in c leads to an increased gap between the emotional distance equilibria of the ambivalent and secure child.

3.5 Slow-time evolution of behavioural parameters. In the previous sections, we find that the different behaviour that is observed in the model (3.10) for different values of the inconsistency parameter a and the insensitivity parameter b are sufficient to distinguish the three different attachment types. We have assumed that the values of these parameters are constant, but in a fuller description, these values will depend on the influence of the environment, and predominantly the actions of the mother toward the child. Some of the fundamental issues in formulating a mathematical model of mother-child attachment centre on understanding how these parameter values are determined and what influences could change them, leading to improvements in the long term development of the child.

Let us assume that while the parameters $(a, b, m, \bar{A}, \cdots)$ may be treated as being fixed over the course a short period, e.g. the length of the strange situation experiment, they can change over sufficiently long times, say over the course of a month. Let $\tau = \epsilon t$ represent a slow timescale, where ϵ is a small parameter. If time is measured in minutes, then for example, ϵ could be one over the number of minutes in a month, $\epsilon \approx 1/43,200 \approx 2 \times 10^{-5}$. If we assume that the parameters, e.g., a, vary only over the course of months, then $a = a(\tau)$, where $a(\tau)$ is assumed to be a smooth function, whose value would essentially be unchanged over a short period of time, i.e., $a(\tau + 60 \times 10^{-5}) \approx a(\tau)$. However, given an extended period over which the mother's response to her child has changed, e.g. due to some intervention, these parameters should evolve in response.

However, it may be useful to assume that the parameter ρ, which appears in the damping term of the nonlinear anxiety model (3.6), varies on intermediate time scales. That is, it can be assumed to be approximately constant on time scales of a single anxious event, e.g. one exit and return of the mother in the strange situation experiment, but that it may vary over the course of several consecutive anxious events, e.g. over the course of the entire experiment. In the strange situation experiment, it has been observed that the mother's ability to regulate the anxiety of the child decreases as the number of absences of the mother increases. Thus, if we assume that, for instance, the parameter ρ is proportional to a windowed time-average of the anxiety A over an interval $[t-\delta t, t]$ i.e,

$$\rho = \rho_s \int_{t-\delta t}^{t} (A - \bar{A}) dt, \tag{3.14}$$

where ρ_s is a constant, then ρ will increase each time the mother leaves the room. The increase in ρ will lead to a slower decay of anxiety. As this rate of decay increases, the child's responses will appear more and more like those of an ambivalent child. See Figure 12. This is consistent with the observation that given several absences of the mother, all children display characteristics of an ambivalent child.

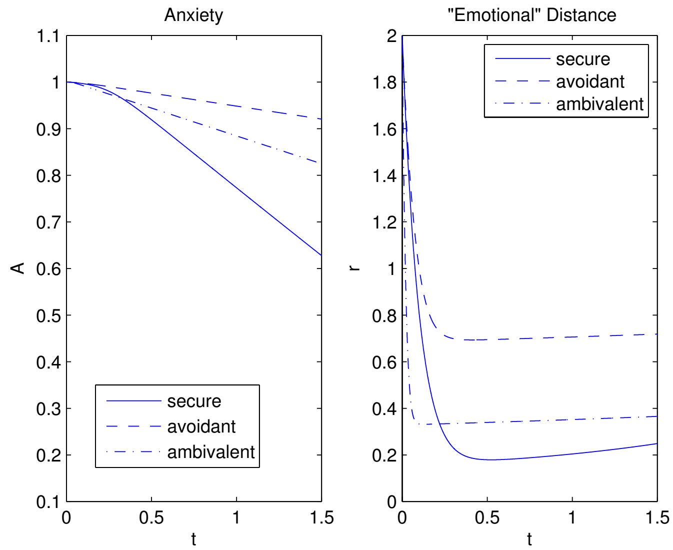

Figure 12 Numerical simulations of the full system with general damping term (i.e. $\rho \neq 0$) as the mother returns to the room. We take $\rho = 20$, while the other parameter values are taken to be the same as those used for Figure 10.

Furthermore, the parameter ρ appears in the damping term multiplied by A^2. Thus, the damping increases as A increases, leading to the realistic situation that the higher the child's anxiety, the more difficult it is to regulate.

3.6 Discussion. The model presented in Section 3 describes the dynamics observed during the strange situation experiment. In particular, using two variables, anxiety A and emotional distance r, the model has the capacity to distinguish between the three types of mother-child interactions observed in the experiment: secure, ambivalent and avoidant. Moreover, the distinction is achieved by varying only two parameters that are directly linked with the mother's parenting style: the mother's insensitivity b and the inconsistency a in her response to her child's needs. In the construction of the model, we chose reasonable representations of how the characteristics associated with these parameters affect the mother's ability to regulate her child's anxiety. The fact that the variation of these two parameters leads to the qualitative distinction of the attachment types supports the claim that these are indeed the most important factors in the problem.

However, at this stage, the model is only phenomenological. The next stage in the model development will be to incorporate quantitative information. In order to do this, we need to find reasonable units for the variables and parameters, and we need to approximate the values of the parameters. In some cases it will likely be necessary to design new experiments specifically for this purpose. We also need to acquire quantitative data sets that provide continuous readings of the model variables throughout an experiment. Anxiety A has been measured using levels of cortisol in the child's saliva. However, existing data sets only include measurements at a few points during an experiment. It may be practical to use heart rate as a measure of anxiety, because it is a quantity that is relatively easy to monitor continuously.

Similarly, data for the emotional distance r is needed. However, before such data can be acquired, it must be determined whether it can be defined in terms of directly measurable quantities such as physical distance, and frequency of visual contact or verbal communication.

The model also contains other parameters, namely "emotional inertia" m, in analogy to Newtonian inertia in the second-order equation describing anxiety, and "curiosity" c, in the equation for emotional distance. At present, we only have a vague concept of the significance of these parameters and it would be interesting to find precise psychological concepts relating to these parameters. This would be particularly significant if those concepts could be quantifiable. By performing various numerical simulations of (3.10), we have found that the qualitative behaviour of the model is not sensitive to these parameters. However, it would be interesting to quantify the model's dependence on these parameters.

4 Control Theory Approach

(This section contributed by Roger Chau).

In this section, we present an outline of how one might take a control theory approach to the problem of mother-child attachment. In particular, a feedback system is proposed to model the interaction between mother and child during stressful situations. The child's perception of the mother's parenting style is modelled using the Preisach model [14], which is commonly used in modelling shape memory alloys (SMAs).

4.1 Feedback System. Consider the feedback system shown in Figure 13. (For background in control theory and feedback systems, please refer to [17]). The system represents the child, whereas the controller, or regulator, models the relationship between the mother and the child. The input disturbance s is the external stress that the child experiences when certain events occur. The reference signal r is the baseline stress level of the child.

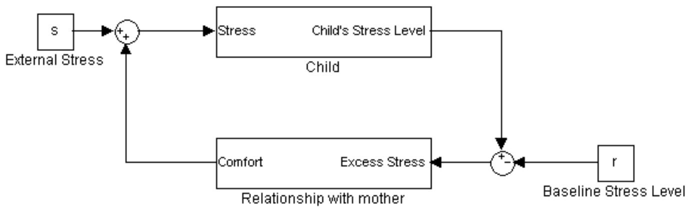

Figure 13 The feedback system

We will first look at the system representing the child; Figure 14 shows the components that make up our *simple child*. We take the input u to the system to be the external stress modulated by the regulator, which represents the relationship of the child with her mother, and we take the output y to be the stress level of the child. In the absence of the mother, the child will amplify the stress received. This is modelled by a gain of k_1 which is the same for every child. In addition, an integrator with gain k_2 is used to model the effect of accumulated stress on the child's stress level.

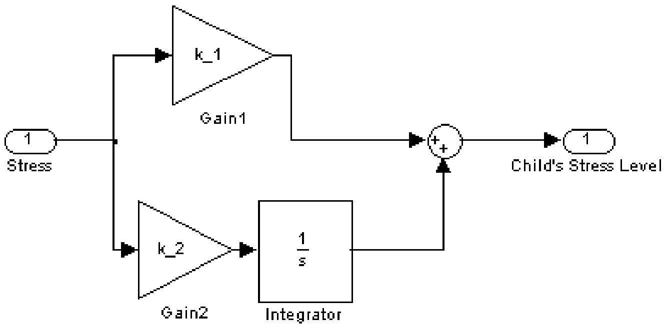

Figure 14 The system representing the child.

The second component of our feedback system is the regulator; see Figure 15. A Proportional-Integral-Differential (PID) controller is used to model the child's relationship with the mother. Note that it takes in the *difference* between the baseline stress level r and the actual stress level of the child y. The regulators job is to reduce this difference or *error* by changing the input to the system. In essence, the proportional gain k_p deals with the present, the integral gain k_i deals with the past, and the differential gain k_d attempts to predict the future. The proportional gain k_p is assumed to be a random value, because in any given situation, the mother's ability to down-regulate the child's anxiety will vary. The integral gain k_i depends on the 'healthiness' of the child's relationship with the mother. If the mother has a healthy relationship with her child, it is easier for her to reduce accumulated stress. Finally, the differential gain k_d depends on the consistency of the mother's parenting style. This has to do with the child's expectation of the mother's behaviour when a stressful situation occurs. Note that the values of these gains may be positive or negative, depending on the child's experience with the mother.

4.2 Preisach Model. While the implementation of the above model seems to be straightforward, it is not clear how some of the parameter values can be determined. Because we are mainly interested in qualitative results, the values of k_1 and k_2 may be taken as unity, and the baseline stress level may be taken as zero. Two parameters that depend on the mother's parenting style are k_i and k_d. These parameters are similar to those used in the other models in this report, and therefore, it is useful to propose a method to derive them. The following is a description of the Preisach model, which is commonly used to model systems with memory.

In [10], a method for mapping human emotions as a continuous surface is proposed. Here, we will follow a similar approach, this time making use of the Preisach model. Consider

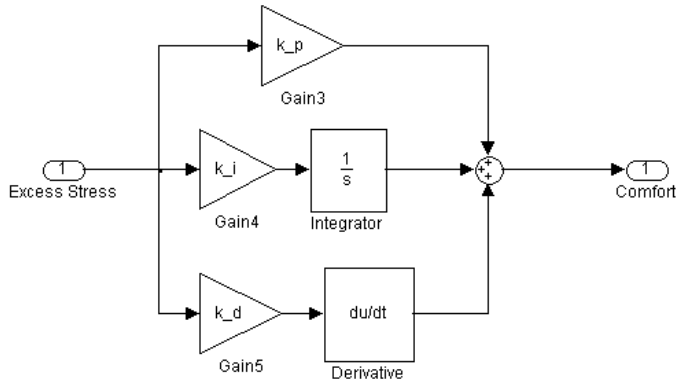

Figure 15 The regulator representing the child's relationship with her mother

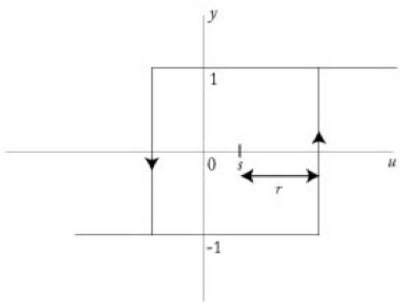

Figure 16 A simple relay

modelling a single emotion with a simple relay as in Figure 16. The input of the relay is the perception or rating of the mother in the child's point of view. The relay has two outputs: +1 corresponding to *good* and -1 corresponding to *bad*. Each relay is categorized by the centre s and the half width $r > 0$.

A relay is used because for a particular emotion to change from one state to another might require different amount of influence. For example, if person A is originally unhappy, then it will take a lot of effort to make A happy; similarly, if A is happy, a lot more *bad influence* is needed to make A unhappy again. Each relay can be uniquely identified by the pair of parameters (r, s), or as ordered pairs in the plane $\mathbb{R}_+ \times \mathbb{R}$. We can set an arbitrary limit on the maximum magnitude of the input. This is possible due to that fact that after a certain threshold, little effects can be observed if we change the input. This restricted domain is the *Preisach Plane* $\mathcal{P}$ shown in Figure 17.

The goal of using the Preisach model is to find a way to quantify the values of k_i and k_d. The output p of the Preisach model can be used to determine the value of k_i. In order to determine k_d, we can keep track of the number of sign changes of the output y over a fixed period of time. There are several programs written for the Preisach model and the PID controller is trivial to implement using, for example, MATLAB.

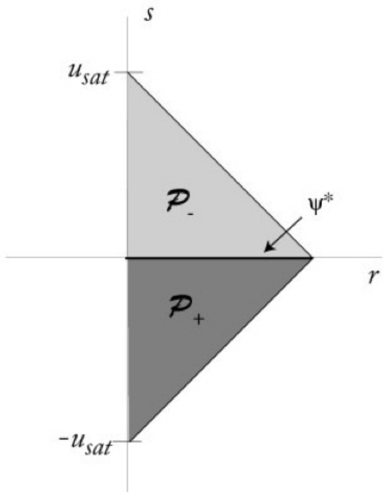

Figure 17 Preisach Plane

5 Summary

In this report, we present three mathematical models relevant to the attachment of a child with her mother. In the first, we approached the problem from a game theory perspective. We find that a simple decision-making process can lead to three distinct attachment types.

In the second model, we present a dynamical system that exhibits various behaviour similar to what might be expected from the three distinct attachment types. We also find that the model parameters that determine the qualitative character of the response are associated with the mother's sensitivity and consistency. In this report, we have only presented qualitative results, and have left some important questions regarding model validation and quantitative predictions unanswered. This will be a subject of future research.

Finally, we present a sketch of a model in control theory. This approach is intriguing, owing to the fact that some of the original results on attachment theory were described in such terms. However, much more work must be done in order to determine whether this model would yield fruitful results.

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References

[1] M.D.S. Ainsworth, M.C. Blehar, E. Waters, and S. Wall. *Patterns of attachment: A psychological study of the strange situation*. Lawrence Erlbaum, Hillsdale, NJ, 1978.

[2] M.D.S. Ainsworth and J. Bowlby. An ethological approach to personality development. *American Psychologist*, 46(4):333–341, 1991.

[3] David P. Barash. *The survival game*. Henry Holt and Company, 2003.

[4] Jessie Bernard. Counseling techniques for arriving at optimum compromises: game- and decision theory. *Marriage and family living*, 21(3):264–274, 1959.

[5] William E. Boyce and Richard C. DiPrima. *Elementary differential equations and boundary value problems*. John Wiley & Sons Inc., New York, 1965.

[6] Steven J. Brams. *The Theory of Moves*. Cambridge University Press, 1994.

[7] Steven J. Brams. Game theory and emotion. *Rationality and Society*, 9(1):91–124, 1997.

[8] Steven J. Brams. Catch-22 and king-of-the-mountain games: Cycling, frustration and power. *Rationality and Society*, 11(2):139–167, 1999.

[9] I. Bretherton. The origins of attachment: John Bowlby and Mary Ainsworth. *Developmental Psychology*, 28(5):759–775, 1992.

[10] E.M. Daly, W.J. Lancee, and J. Polivy. A conical model for the taxonomy of emotional experience. *Journal of Personality and Social Psychology*, 45(2):443–457, 1983.

[11] G. F. Franklin, J. D. Powell, and A. Emami-Naeini. *Feedback control of dynamic systems*. Addison-Wesley, Reading MA, 1988.

[12] F.R. Gantmacher. *Matrix Theory. Vol. II*. AMS, Chelsea, 2000.

[13] Arjita Ghosh and Sandip Sen. Theory of moves learners: Towards non-myopic equilibria. In *Proceedings of the fourth international joint conference on Autonomous agents and multiagent systems*, pages 74–80. ACM Press, 2005.

[14] R.B. Gorbet. *Control of Hysteretic Systems with Preisach Representations*. PhD thesis, University of Waterloo, Ontario, Canada, 1997.

[15] R. Duncan Luce and Howard Raiffa. *Games and Decisions: Introduction and critical survey*. Dover, 1989.

[16] J. McKinsey. *Introduction to the Theory of Games*. The Rand Series. McGraw-Hill Book Company, 1952.

[17] K.A. Morris. *Introduction to Feedback Control*. Harcourt/Academic Press, San Diego, CA, 2001.

[18] John von Neumann and Oskar Morgenstern. *The Theory of Games and Economic Behavior*. Princeton University Press, 1944.