Options on Baskets

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Section 1: Introduction

This report describes progress made during MPI 2000 on the problem presented by Pat Hagan, then from Numerix, namely to devise fast, accurate methods to determine the value of options on baskets as well as the correct associated hedges. The method employed is a perturbation procedure based on a *small volatility* assumption.

This report is organized as follows: In Section 2 we give some background on the derivation of a Black-Scholes type equation for the value of options on baskets; in Section 3 a general perturbation expansion is introduced based on a small parameter measuring volatility; in Section 4 the method is applied to European call and put options, and to other options that can be easily treated following those two *vanilla options*; Section 5 treats digital options; in Section 6 barrier options are discussed; numerical simulation results based on the general expansion procedure are presented in Section 7, as well as comparison of the calculations based on the expansion procedure with numerical calculations for the case of 1, 2 and 3 assets in the basket; and finally, Section 8 presents some conclusions and directions for further research.

Section 2: Deriving the Deterministic Equation

We wish to derive a deterministic equation for the price of an option on a *basket* (collection) of n assets. It is assumed that each of the asset (usually stock) prices S_k follows a geometric Brownian motion with drift coefficient $\tilde{\mu}_k$ and variance parameter (volatility) σ_k. (Here σ_k is considered known and it will be shown that the value of $\tilde{\mu}_k$ will be irrelevant in our analysis.) Thus

$$dS_k = \tilde{\mu}_k S_k \, dt + \sigma_k S_k \, dW_k, \tag{2.1}$$

where dW_k is a Wiener process.

Since we are considering assets priced in other currencies, we must also consider the foreign exchange rate A_k, defined as the number of units of a base currency (subscript zero) per unit of currency k, which is modeled in the same way as the asset price:

$$dA_k = \nu_k A_k \, dt + b_k A_k \, dZ_k, \tag{2.2}$$

where Z_k is a Wiener process. Again ν_k is unknown and σ_k is considered known. Clearly these variables may be correlated. Therefore, we define the correlations as follows, which are simply rules of thumb:

$$dW_k \, dZ_k = \rho_k \, dt, \qquad dW_j \, dW_k = \rho_{jk} \, dt. \tag{2.3}$$

Clearly $\rho_{jj} = 1$. Note that we have assumed that the foreign exchange rates are independent of one another. This assumption can be relaxed.

We now wish to determine the underlying probability distribution for S_k. We present some heuristic arguments; more details may be found in [1]. We note from (2.3) that dW_k and dZ_k are roughly $O(\sqrt{dt})$. Therefore, we have that

$$df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S_k} \, dS_k + \frac{1}{2} \frac{\partial^2 f}{\partial S_k^2} \, dS_k \, dS_k + o(dt)$$

$$df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S_k} \, dS_k + \frac{1}{2} \frac{\partial^2 f}{\partial S_k^2} (\sigma_k S_k)(\sigma_k S_k) \, dW_k \, dW_k + o(dt) \tag{2.4a}$$

$$= \left(\frac{\partial f}{\partial t} + \tilde{\mu}_k S_k \frac{\partial f}{\partial S_k} + \frac{\sigma_k^2 S_k^2}{2} \frac{\partial^2 f}{\partial S_k^2} \right) dt + \frac{\partial f}{\partial S_k} \sigma_k S_k \, dW_k \tag{2.4b}$$

for any function $f(S_k, t)$. Now let $f = \log S_k$. Then (2.4b) becomes

$$d(\log S_k) = \left[\tilde{\mu}_k S_k \frac{1}{S_k} + \frac{\sigma_k^2 S_k^2}{2} \left(-\frac{1}{S_k^2} \right) \right] dt + \frac{1}{S_k} \sigma_k S_k \, dW_k = \left(\tilde{\mu}_k - \frac{\sigma_k^2}{2} \right) dt + \sigma_k \, dW_k.$$

Thus, $\log S_k$ follows a normal distribution with standard deviation $\sqrt{t}\,\sigma_k$ and mean $\tilde{\mu}_k - \sigma_k^2/2$. Hence, we call the probability distribution that S_k follows the *lognormal distribution.*

We denote the price of a European option (valued in the base currency) by $\tilde{v}$. Consider a portfolio π composed of one option and "short" (that is, owing) λ_k units of asset k and q_k units of currency k for each of the n assets in the basket. At any time t, the value of this portfolio (in base currency) is

$$\pi(\mathbf{S}(t), t) = \tilde{v}(\mathbf{S}, t) - \sum_{k=1}^{n}(\lambda_k S_k A_k + q_k A_k), \tag{2.5}$$

where we use bold face to indicate a vector (hence $\mathbf{S}$ is the vector of the S_k). Note that in the above the value of the portfolio and the option do not depend on $\mathbf{A}$ explicitly. This fact can be verified, but we do not do so here.

We next wish to find the change $d\pi$ in the value of the portfolio during the next instant dt in time. By extending the result in (2.4a) to multiple variables, we immediately obtain

$$df = \frac{\partial f}{\partial t}\,dt + \sum_{k=1}^{n}\frac{\partial f}{\partial S_k}\,dS_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 f}{\partial S_j \partial S_k}\rho_{jk}\sigma_j S_j \sigma_k S_k\,dt + o(dt), \tag{2.6}$$

for any function $f(\mathbf{S}, t)$. Here the ρ_{jk} term arises from (2.3). Using (2.6) repeatedly (sometimes using A_j instead of S_j as the independent variable), we see that in the next dt the value of the portfolio changes by

$$\begin{aligned} d\pi = {}& \frac{\partial \tilde{v}}{\partial t}\,dt + \sum_{k=1}^{n}\frac{\partial \tilde{v}}{\partial S_k}\,dS_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \tilde{v}}{\partial S_j \partial S_k}\rho_{jk}\sigma_j S_j \sigma_k S_k\,dt - \sum_{k=1}^{n}(\lambda_k S_k + q_k)\,dA_k \\ & - \sum_{k=1}^{n}\lambda_k A_k\,dS_k - \sum_{k=1}^{n}\lambda_k \sigma_k S_k b_k A_k \rho_k\,dt - \text{cost of borrowing}, \end{aligned} \tag{2.7a}$$

where the next-to-last term arises from the last term in (2.6), except now we are taking a mixed partial (the only second derivative that survives) with respect to S_j and A_j. The cost of borrowing is given by

$$\text{cost of borrowing} = \sum_{k=1}^{n} q_k A_k r_k\,dt, \tag{2.7b}$$

where r_k is the risk-free interest rate for currency k. Combining equations (2.7), we obtain

$$\begin{aligned} d\pi = {}& \left[\frac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \tilde{v}}{\partial S_j \partial S_k}\rho_{jk}\sigma_j S_j \sigma_k S_k - \sum_{k=1}^{n}(q_k A_k r_k + \lambda_k \sigma_k S_k b_k A_k \rho_k)\right] dt \\ & - \sum_{k=1}^{n}(\lambda_k S_k + q_k)\,dA_k + \sum_{k=1}^{n}\left(\frac{\partial \tilde{v}}{\partial S_k} - \lambda_k A_k\right) dS_k. \end{aligned} \tag{2.8}$$

Equation (2.8) is stochastic only because of the last two terms. However, we note that the number of units λ_k and the amount of currency q_k are *at our disposal* and hence can be chosen to be anything. Therefore, we choose

$$\lambda_k = \frac{1}{A_k}\frac{\partial \tilde{v}}{\partial S_k}, \qquad q_k = -\lambda_k S_k = -\frac{S_k}{A_k}\frac{\partial \tilde{v}}{\partial S_k}. \tag{2.9}$$

to zero out the stochastic terms (*i.e.*, to eliminate risk). This technique is called *dynamic* Δ-*hedging*, since it requires that we continually add and subtract shares and currencies from our portfolio depending on $\partial\tilde{v}/\partial S_k$. We note that we have now made an error, since in the above analysis we assumed that both q_k and λ_k were constant, not functions of time. This is the *Black-Scholes* error, but merely adds lower-order corrections to the terms above.

Substituting (2.9) into (2.5) and (2.8), we obtain

$$\pi(\mathbf{S}, t) = \tilde{v}(\mathbf{S}, t), \tag{2.10a}$$

which implies the net contribution in value to the portfolio from the hedge positions is zero. (Note this is different from the standard Black-Scholes case.) Taking the change in value over a small time frame dt, we obtain

$$d\pi = \left[\frac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \tilde{v}}{\partial S_j \partial S_k}\rho_{jk}\sigma_j S_j \sigma_k S_k + \sum_{k=1}^{n} S_k(r_k - \sigma_k b_k \rho_k)\frac{\partial \tilde{v}}{\partial S_k}\right] dt. \tag{2.10b}$$

We now assume that the market is *arbitrage-free*; that is, no one can set up a situation where he or she can *always* make money without risk. Thus, $d\pi$ must be exactly the same whether we buy an option or whether we invest the money at the risk-free rate r_0 of the base currency. Therefore, using this fact and in (2.10), we obtain

$$r_0 \pi dt = r_0 \tilde{v} dt = \left[\frac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \tilde{v}}{\partial S_j \partial S_k}\rho_{jk}\sigma_j S_j \sigma_k S_k + \sum_{k=1}^{n}(r_k - \sigma_k b_k \rho_k) S_k\frac{\partial \tilde{v}}{\partial S_k}\right] dt. \tag{2.11}$$

Thus,

$$\frac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \tilde{v}}{\partial S_j \partial S_k}\rho_{jk}\sigma_j S_j \sigma_k S_k + \sum_{k=1}^{n}(r_k - \sigma_k b_k \rho_k) S_k\frac{\partial \tilde{v}}{\partial S_k} = r_0 \tilde{v}, \qquad t \in [0, t_{\text{ex}}], \tag{2.12}$$

where t_{ex} is the *exercise date* at which the option will pay off. For the purposes of this report, we assume that b_k, $\tilde{\mu}_k$, ρ_k, ρ_{jk}, σ_k, and ν_k are independent of t.

From the Feynman-Kac theorem [2], [3], it can be shown that the solution $\tilde{v}(t)$ of (2.12) is the following expected value:

$$\tilde{v}(t) = e^{-r_0(t_{\text{ex}}-t)} E\left[\tilde{v}(\mathbf{S}(t_{\text{ex}}), t_{\text{ex}}) | \mathbf{S}(t)\right],$$

under the probability distribution given by

$$dS_k = (r_k - \sigma_k b_k \rho_k) S_k\, dt + \sigma_k S_k\, d\tilde{W}_k, \tag{2.13a}$$

$$d\tilde{W}_j\, d\tilde{W}_k = \rho_{jk}\, dt, \tag{2.13b}$$

where $d\tilde{W}_k$ is Brownian motion. Thus, the value of the option given by the solution of (2.12) is simply the discounted expected value in a world in which the probability distributions evolve according to (2.13), instead of the real-world probabilities (2.1). This "new" world is known as the *risk-neutral* world, and apart from the discount factor, the option price is a martingale in this world. This is a reflection of the fundamental theorem of arbitrage-free pricing.

We now perform several substitutions to simplify (2.12). First, due to the similarities between (2.1) and (2.13b), we *define* the new parameter μ_k as follows:

$$\mu_k = r_k - \sigma_k b_k \rho_k. \tag{2.14}$$

Note that in contrast to $\tilde{\mu}_k$, μ_k is a known quantity, as each of its component parts are known. Also, μ_k is not the true drift coefficient for the asset price. It is simply the drift coefficient for a new random process that causes the option price to behave as a martingale.

Next, conditions in the financial problem are given at t_{ex}, which is a *final* condition. This is fine, as (2.12) is a *backwards* heat equation. However, for simplicity, we introduce the time to expiry, τ, and the future value, v, as follows:

$$\tilde{v}(\mathbf{S}, t) = e^{-r_0 \tau} v(\mathbf{S}, \tau), \qquad \tau = t_{\text{ex}} - t. \tag{2.15}$$

Substituting (2.15) into (2.12), we obtain

$$\frac{\partial v}{\partial \tau} = \sum_{k=1}^{n} \mu_k S_k \frac{\partial v}{\partial S_k} + \frac{1}{2} \sum_{j=1}^{n} \sigma_j S_j \sum_{k=1}^{n} \rho_{jk} \sigma_k S_k \frac{\partial^2 v}{\partial S_j \partial S_k}, \quad \tau \in [0, t_{\text{ex}}]. \tag{2.16}$$

We construct a basket of the assets, giving each a weight w_k in the basket. Therefore, the value S of the basket is given by

$$S = \sum_{k=1}^{n} w_k S_k. \tag{2.17}$$

The option pays off depending on how S compares with K, the *strike price*, which in simple options is where the option begins to pay off. Note that the strike price is *for the entire basket*. Therefore, V depends on the asset price only through the combination $w_k S_k$. We also note that (2.16) is equidimensional in S_k, so such a substitution may be made without difficulty. In addition, we wish to get rid of the drift term, so we let

$$v(\mathbf{S}, \tau) = \tilde{V}(\mathbf{x}, \tau), \qquad x_k = e^{\mu_k \tau} w_k S_k. \tag{2.18}$$

Here x_k is a weighted future price of the asset. Substituting (2.18) into (2.16), we obtain

$$\frac{\partial \tilde{V}}{\partial \tau} = \frac{1}{2}\sum_{j=1}^{n} \sigma_j x_j \sum_{k=1}^{n} \rho_{jk}\sigma_k x_k \frac{\partial^2 \tilde{V}}{\partial x_j \partial x_k}. \tag{2.19}$$

Note that this substitution eliminates the drift term from the equation by eliminating the growth rate of each of the individual asset prices.

We now proceed to the *small vol* limit by assuming that each of the volatilities is small. Combining terms for simplicity, we let

$$\rho_{jk}\sigma_j\sigma_k = \epsilon^2 z_{jk}, \quad z_{jk} = O(1), \quad 0 < \epsilon \ll 1. \tag{2.20}$$

(This is equivalent to looking at a short-time limit.) Thus (2.19) becomes

$$\frac{\partial \tilde{V}}{\partial \tau} = \frac{\epsilon^2}{2}\sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \frac{\partial^2 \tilde{V}}{\partial x_j \partial x_k}, \tag{2.21}$$

and to leading order we see that the value of the option will not change. This makes sense, as low volatility implies that the basket price will not change much from its value now to its value at *expiry* (the exercise date).

However, (2.21) is a diffusion equation, and hence we see that any discontinuities in $\tilde{V}(\mathbf{x}, 0)$ or its derivatives must be smoothed in a layer about the point of discontinuity. In many cases, this discontinuity occurs at the strike price K. We note from (2.18) that at expiry $\tau = 0$,

$$S = \sum_{j=1}^{n} x_j.$$

Therefore, we introduce the following transformation:

$$\tilde{V}(\mathbf{x}, \tau) = V(\mathbf{x}, \zeta, \tau), \qquad \zeta = \frac{1}{\epsilon}\left(\sum_{j=1}^{n} x_j - K\right). \tag{2.22}$$

Note from (2.22) that technically we have added another degree of freedom, since we have not replaced any variables. If we replaced x_1, say, with ζ, we would have to write x_1 in terms of K, the other x_j, and ζ. To eliminate this algebraic complexity, we simply add the variable ζ and keep $S = K + O(\epsilon)$ as a constraint.

Substituting (2.22) into (2.21), we obtain

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \left(\frac{\partial}{\partial \zeta} + \epsilon\frac{\partial}{\partial x_j}\right)\left(\frac{\partial}{\partial \zeta} + \epsilon\frac{\partial}{\partial x_k}\right) V. \tag{2.23}$$

Lastly, we consider the boundary and initial conditions. First, we must match to the outer solution:

$$V(\mathbf{x}, -\infty, \tau) = \tilde{V}(S = K^-, \tau), \qquad V(\mathbf{x}, \infty, \tau) = \tilde{V}(S = K^+, \tau). \tag{2.24a}$$

For our initial condition, we note by our assumption that V must vary on the ζ scale initally, so we have

$$V(\mathbf{x}, \zeta, 0) = p(\mathbf{x}, \zeta), \tag{2.24b}$$

where p is the *payoff function* that describes what the investor will receive as a function of the asset prices.

Section 3: General Perturbation Expansion

We now introduce a perturbation expansion in V:

$$V(\mathbf{x}, \zeta, \tau; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j V_j(\mathbf{x}, \zeta, \tau). \tag{3.1}$$

Substituting (3.1) into (2.23) and expanding to leading three orders, we obtain

$$\frac{\partial (V_0 + \epsilon V_1 + \epsilon^2 V_2)}{\partial \tau} = \frac{1}{2} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \left(\frac{\partial}{\partial \zeta} + \epsilon \frac{\partial}{\partial x_j} \right) \left(\frac{\partial}{\partial \zeta} + \epsilon \frac{\partial}{\partial x_k} \right) (V_0 + \epsilon V_1 + \epsilon^2 V_2).$$

Working initially with the $O(1)$ equation, we obtain

$$\frac{\partial V_0}{\partial \tau} - \frac{\alpha}{2} \frac{\partial^2 V_0}{\partial \zeta^2} \equiv \mathcal{L} V_0 = 0, \tag{3.2a}$$

$$\alpha = \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k. \tag{3.2b}$$

We note from (3.2) that $\mathbf{x}$ enters into the equation only through the coefficent α, and hence our solution depends on $\mathbf{x}$ only parametrically. In other words, to leading order the value of the option behaves like the value of the option on a single asset, whose price is the weighted average of the prices of all assets in the basket. In addition, we note that we could eliminate α from (3.2a) simply by scaling τ. Therefore, V_0 can depend on τ only through the combination $\alpha\tau$, so we have that

$$V_0 = V_0(\zeta, \alpha\tau) \qquad \Longrightarrow \qquad \alpha^j \frac{\partial^j V_0}{\partial \alpha^j} = \tau^j \frac{\partial^j V_0}{\partial \tau^j}. \tag{3.3}$$

Proceeding to the $O(\epsilon)$ equation, we obtain

$$\frac{\partial V_1}{\partial \tau} - \frac{\alpha}{2} \frac{\partial^2 V_1}{\partial \zeta^2} = \frac{1}{2} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right) \frac{\partial V_0}{\partial \zeta},$$

where the right-hand side term arises from the cross-terms in the derivatives. Since V_0 depends on $\mathbf{x}$ only through α, we may rewrite the above as

$$\frac{\partial V_1}{\partial \tau} - \frac{\alpha}{2} \frac{\partial^2 V_1}{\partial \zeta^2} = \frac{1}{2} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \left(\frac{\partial \alpha}{\partial x_j} + \frac{\partial \alpha}{\partial x_k} \right) \frac{\partial^2 V_0}{\partial \alpha \partial \zeta}$$

$$
\begin{aligned}
&= \frac{\tau}{2\alpha} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \left(\frac{\partial \alpha}{\partial x_j} + \frac{\partial \alpha}{\partial x_k} \right) \frac{\partial^2 V_0}{\partial \tau \partial \zeta} \\
&= 2\gamma\tau \frac{\partial^2 V_0}{\partial \tau \partial \zeta}, \qquad (3.4a)
\end{aligned}
$$

$$
\gamma = \frac{1}{4\alpha} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k \left(\frac{\partial \alpha}{\partial x_j} + \frac{\partial \alpha}{\partial x_k} \right). \qquad (3.4b)
$$

We solve the above by means of a clever trick. We note that if V_0 satisfies (3.2a), so does any derivative of V_0. Therefore, motivated by the form of the right-hand side, we assume a solution of the following form:

$$
V_{1,\mathrm{p}} = f_1(\tau) \frac{\partial^2 V_0}{\partial \tau \partial \zeta}. \qquad (3.5)
$$

Substituting (3.5) into (3.4a), we obtain

$$
\begin{aligned}
f_1' \frac{\partial^2 V_0}{\partial \tau \partial \zeta} + f_1 \mathcal{L} \frac{\partial^2 V_0}{\partial \tau \partial \zeta} &= 2\gamma\tau \frac{\partial^2 V_0}{\partial \tau \partial \zeta} \\
f_1' &= 2\gamma\tau \\
f_1(\tau) &= \gamma\tau^2 + \text{constant} \\
V_{1,\mathrm{p}}(\zeta, \tau) &= \gamma\tau^2 \frac{\partial^2 V_0}{\partial \tau \partial \zeta}, \qquad (3.6)
\end{aligned}
$$

where we have taken the constant equal to zero because this yields a homogeneous solution. Note that the α-dependence is not as simple as in (3.3), and that V_1 depends on x parametrically through both α and γ.

Lastly, we turn to the $O(\epsilon^2)$ equation:

$$
\begin{aligned}
\mathcal{L} V_2 &= \frac{1}{2} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk} x_k Q_{jk}, \\
Q_{jk} &= \frac{\partial^2 V_0}{\partial x_j x_k} + \left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right) \frac{\partial V_1}{\partial \zeta} \\
&= \left(\frac{\partial \alpha}{\partial x_j} \right) \left(\frac{\partial \alpha}{\partial x_k} \right) \frac{\partial^2 V_0}{\partial \alpha^2} + \frac{\partial^2 \alpha}{\partial x_j \partial x_k} \frac{\partial V_0}{\partial \alpha} + \left(\frac{\partial \alpha}{\partial x_j} + \frac{\partial \alpha}{\partial x_k} \right) \frac{\partial^2 V_1}{\partial \alpha \partial \zeta} \\
&\quad + \left(\frac{\partial \gamma}{\partial x_j} + \frac{\partial \gamma}{\partial x_k} \right) \frac{\partial^2 V_1}{\partial \zeta \partial \gamma}. \qquad (3.7)
\end{aligned}
$$

At this stage we make a further simplification which **NEED NOT WORK IN EVERY CASE**, but which works for most of the examples we consider here. (A case which may fail appears in section 6.) We assume that no homogeneous solution is needed for V_1, so

$V_1 = V_{1,\mathrm{p}}$. Using this fact, (3.2a), (3.3), and (3.6) in (3.7), we obtain

$$Q_{jk} = \frac{\tau^2}{\alpha^2}\left(\frac{\partial\alpha}{\partial x_j}\right)\left(\frac{\partial\alpha}{\partial x_k}\right)\frac{\partial^2 V_0}{\partial\tau^2} + \frac{\tau}{\alpha}\frac{\partial^2\alpha}{\partial x_j\partial x_k}\frac{\partial V_0}{\partial\tau} + \gamma\tau^2\left(\frac{\partial\alpha}{\partial x_j}+\frac{\partial\alpha}{\partial x_k}\right)\frac{\partial^4 V_0}{\partial\alpha\partial\tau\partial\zeta^2}$$
$$+\tau^2\left(\frac{\partial\gamma}{\partial x_j}+\frac{\partial\gamma}{\partial x_k}\right)\frac{\partial^3 V_0}{\partial\zeta^2\partial\tau}$$
$$= \frac{\tau^2}{\alpha^2}\left(\frac{\partial\alpha}{\partial x_j}\right)\left(\frac{\partial\alpha}{\partial x_k}\right)\frac{\partial^2 V_0}{\partial\tau^2} + \frac{\tau}{\alpha}\frac{\partial^2\alpha}{\partial x_j\partial x_k}\frac{\partial V_0}{\partial\tau} + \frac{2\gamma\tau^3}{\alpha^2}\left(\frac{\partial\alpha}{\partial x_j}+\frac{\partial\alpha}{\partial x_k}\right)\frac{\partial^3 V_0}{\partial\tau^3}$$
$$+\frac{2\tau^2}{\alpha}\left(\frac{\partial\gamma}{\partial x_j}+\frac{\partial\gamma}{\partial x_k}\right)\frac{\partial^2 V_0}{\partial\tau^2},$$

$$\mathcal{L}V_2 = 4Q_3\tau^3\frac{\partial^3 V_0}{\partial\tau^3} + 3Q_2\tau^2\frac{\partial^2 V_0}{\partial\tau^2} + 2Q_1\tau\frac{\partial V_0}{\partial\tau}, \tag{3.8}$$

$$Q_1 = \frac{1}{4\alpha}\sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk}x_k\frac{\partial^2\alpha}{\partial x_j\partial x_k}, \tag{3.9a}$$

$$Q_2 = \frac{1}{6\alpha}\sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk}x_k\left[\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial x_j}\right)\left(\frac{\partial\alpha}{\partial x_k}\right) + 2\left(\frac{\partial\gamma}{\partial x_j}+\frac{\partial\gamma}{\partial x_k}\right)\right], \tag{3.9b}$$

$$Q_3 = \frac{\gamma}{4\alpha^2}\sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk}x_k\left(\frac{\partial\alpha}{\partial x_j}+\frac{\partial\alpha}{\partial x_k}\right) = \frac{\gamma}{4\alpha^2}(4\alpha\gamma) = \frac{\gamma^2}{\alpha}. \tag{3.9c}$$

Simplification of all these parameters is performed in Appendix A.

Substituting (3.9c) into (3.8), we obtain

$$\frac{\partial V_2}{\partial\tau} - \frac{\alpha}{2}\frac{\partial^2 V_2}{\partial\zeta^2} = \frac{4\gamma^2}{\alpha}\tau^3\frac{\partial^3 V_0}{\partial\tau^3} + 3Q_2\tau^2\frac{\partial^2 V_0}{\partial\tau^2} + 2Q_1\tau\frac{\partial V_0}{\partial\tau}. \tag{3.10}$$

Using the same technique as in (3.5) and (3.6), we easily obtain that

$$V_{2,\mathrm{p}}(\zeta,\tau) = \frac{\gamma^2}{\alpha}\tau^4\frac{\partial^3 V_0}{\partial\tau^3} + Q_2\tau^3\frac{\partial^2 V_0}{\partial\tau^2} + Q_1\tau^2\frac{\partial V_0}{\partial\tau}. \tag{3.11}$$

Though at this stage we have not introduced boundary conditions, we do wish to make a brief remark. If we examine equation (2.21) for the outer solution $\tilde{V}$, we see that if we expand $\tilde{V}$ in a series as in (3.1), the first correction will come at $O(\epsilon^2)$. Therefore, special care must be taken when matching V_2 if we have a solution where $\tilde{V}_2 \neq 0$. (This will not occur in this manuscript, but should be mentioned.)

We note that once we have obtained our solution, the Δ-hedges λ_k and q_k needed at each stage may be calculated from (2.9).

Perturbing Parameters

Though a perturbation solution of the form described by (3.2a), (3.6), and (3.11) would solve the problem, it would not be in the most convenient form. The forms of the expressions lead us to believe that Gaussian distributions, their integrals, and their derivatives will be involved in our expressions. Evaluating them in each of the three solutions over multiple time and spatial steps (to establish hedges) would be time-consuming. Therefore, what we would like to do is establish an equivalent solution via a renormalization as follows:

$$V_0(\zeta,\tau;\alpha)+\epsilon V_1(\zeta,\tau;\alpha)+\epsilon^2 V_2(\zeta,\tau;\alpha)=V_0(\zeta,\tau;\alpha+\epsilon\alpha_1+\epsilon^2\alpha_2)+O(\epsilon^3)\,. \tag{3.12}$$

In other words, we replace the perturbation series in V with a perturbation series in α.

Here we have suppressed the dependence of V_1 and V_2 on γ and the Q_j. If we can solve for the α_j, then we need perform only one Gaussian computation, while performing simple multiplies to obtain the perturbed parameter. The perturbed parameter can then be interpreted in terms of various approximations made in the problem (see section 4).

To solve for the α_i, we equate like powers of ϵ:

$$\begin{aligned}V_0(\zeta,\tau;\alpha)+\epsilon V_1(\zeta,\tau;\alpha)+\epsilon^2 V_2(\zeta,\tau;\alpha) &= V_0(\zeta,\tau;\alpha)+\epsilon\alpha_1\frac{\partial V_0}{\partial\alpha}(\zeta,\tau;\alpha)\\ &\quad+\epsilon^2\alpha_2\frac{\partial V_0}{\partial\alpha}(\zeta,\tau;\alpha)+\frac{\epsilon^2\alpha_1^2}{2}\frac{\partial^2 V_0}{\partial\alpha^2}(\zeta,\tau;\alpha)+O(\epsilon^3)\,.\end{aligned}$$

Dropping the arguments for simplicity, we obtain

$$\alpha_1 = V_1\left(\frac{\partial V_0}{\partial\alpha}\right)^{-1}, \tag{3.13a}$$

$$\alpha_2 = \left(V_2-\frac{\alpha_1^2}{2}\frac{\partial^2 V_0}{\partial\alpha^2}\right)\left(\frac{\partial V_0}{\partial\alpha}\right)^{-1}. \tag{3.13b}$$

Since we would like to obtain such parameter dependence no matter the initial condition, at this stage we try to establish formulas for the α_j in the general case. Therefore, we must solve (3.2a) subject to a general initial condition, given by (2.24b):

$$V_0(\zeta,0)=p(\zeta), \tag{3.14}$$

where we now assume (as will often be the case) that p depends on $\mathbf{x}$ only through ζ. The solution to this problem is well-known [4]; it is given by

$$\begin{aligned}V_0(\zeta,\tau) &= \frac{1}{\sqrt{2\alpha\pi\tau}}\int_{-\infty}^{\infty}p(\zeta')\exp\left(-\frac{(\zeta-\zeta')^2}{2\alpha\tau}\right)d\zeta'\\ &= \int_{-\infty}^{\infty}p(\zeta')\mathcal{K}(u')\,d\zeta',\end{aligned} \tag{3.15a}$$

$$\mathcal{K}(u')=\frac{\mathcal{G}(u')}{\sqrt{\alpha\tau}}, \tag{3.15b}$$

where $\mathcal{G}(u')$ is the Gaussian distribution, given by

$$\mathcal{G}(u')=\frac{e^{-u'^2/2}}{\sqrt{2\pi}}, \tag{3.16a}$$

and u' is defined by

$$u' = \frac{\zeta - \zeta'}{\sqrt{\alpha\tau}}. \tag{3.16b}$$

Using the general differentiation formula (B.4) from Appendix B, we obtain

$$\begin{aligned}
\frac{\partial V_0}{\partial \alpha} &= \frac{\tau}{\alpha}\frac{\partial V_0}{\partial \tau} = \frac{\tau}{\alpha}\int_{-\infty}^{\infty} p(\zeta')\frac{\partial \mathcal{K}(u')}{\partial \tau}\,d\zeta' = \frac{1}{4\tau}\frac{\tau}{\alpha}\int_{-\infty}^{\infty} p(\zeta')H_2\left(\frac{u'}{\sqrt{2}}\right)\mathcal{K}(u')\,d\zeta' \\
&= \frac{1}{4\alpha}\int_{-\infty}^{\infty} p(\zeta')H_2\left(\frac{u'}{\sqrt{2}}\right)\mathcal{K}(u')\,d\zeta', \qquad (3.17a) \\
V_1 &= \gamma\tau^2\frac{\partial^2 V_0}{\partial\tau\partial\zeta} = \gamma\tau^2\int_{-\infty}^{\infty} p(\zeta')\frac{\partial^2\mathcal{K}(u')}{\partial\tau\partial\zeta}\,d\zeta' \\
&= \gamma\tau^2\int_{-\infty}^{\infty} p(\zeta')\left(-\frac{1}{4\tau\sqrt{2\alpha\tau}}\right)H_3\left(\frac{u'}{\sqrt{2}}\right)\mathcal{K}(u')\,d\zeta' \\
&= -\frac{\gamma}{4}\sqrt{\frac{\tau}{2\alpha}}\int_{-\infty}^{\infty} p(\zeta')H_3\left(\frac{u'}{\sqrt{2}}\right)\mathcal{K}(u')\,d\zeta', \qquad (3.17b)
\end{aligned}$$

where $H_j(\cdot)$ is the jth Hermite polynomial.

There is no simple relationship between equations (3.17), and thus this computation must be done on a case-by-case basis.

Section 4: Basic Options

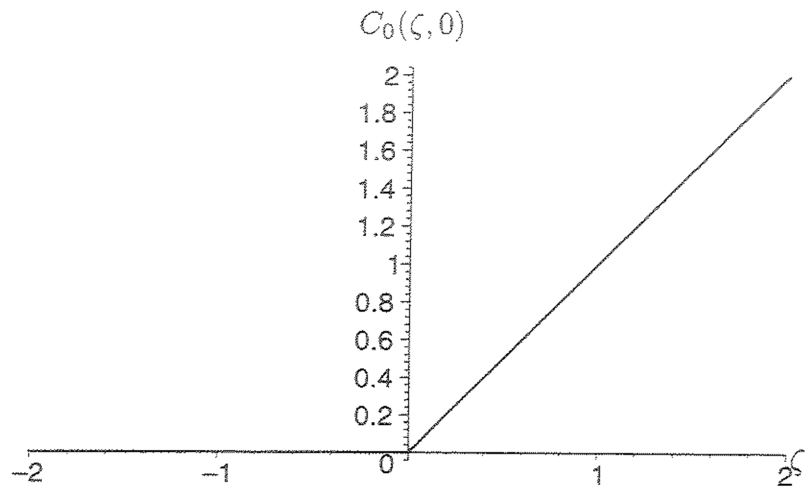

Figure 4.1. Call option payoff.

We begin with the simple case of a European call option. Here the payoff is given by

$$p_C(\mathbf{x}) = [S - K]^+ = \begin{cases} 0, & S < K, \\ S - K, & S > K. \end{cases} \tag{4.1}$$

Therefore, we have a discontinuity in the first derivative at $S = K$. Rewriting the payoff function in the ζ-coordinates, we have

$$p_C(\zeta) = [\epsilon\zeta]^+ = \begin{cases} 0, & \zeta < 0, \\ \epsilon\zeta, & \zeta > 0. \end{cases} \tag{4.2}$$

We note that this behavior will give rise to a corner layer, so we let

$$V_{j+1}(\zeta, \tau) = C_j(\zeta, \tau), \quad j > 0, \tag{4.3}$$

where the C denotes the call option. Thus $V_0 \equiv 0$ and $C_0(\zeta, 0) = [\zeta]^+$, as shown in Figure 4.1.

Substituting $p(\zeta') = [\zeta']^+$ into (3.15), we obtain

$$\begin{aligned} C_0(\zeta, \tau) &= \int_0^\infty \frac{\zeta'}{\sqrt{\alpha\tau}} \mathcal{G}(u')\, d\zeta' = \int_0^\infty \left(\frac{\zeta}{\sqrt{\alpha\tau}} - u' \right) \mathcal{G}(u')\, d\zeta' \\ &= \zeta \int_u^{-\infty} \mathcal{G}(u')(-du') - \sqrt{\alpha\tau} \int_u^{-\infty} u' \mathcal{G}(u')(-du'), \end{aligned} \tag{4.4a}$$

$$u = \frac{\zeta}{\sqrt{\alpha\tau}}. \tag{4.4b}$$

Continuing to simplify (4.4a), we obtain

$$C_0(\zeta,\tau) = \zeta\mathcal{N}(u) - \sqrt{\alpha\tau}\int_{-\infty}^{u} u'\mathcal{G}(u')\,du', \tag{4.5a}$$

$$\mathcal{N}(u) = \int_{-\infty}^{u} \mathcal{G}(u')\,du' = 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{u}{\sqrt{2}}\right). \tag{4.5b}$$

Note from (4.5b) that

$$\mathcal{N}'(u) = \mathcal{G}(u). \tag{4.6a}$$

To complete our solution, we need the following identity:

$$\mathcal{G}'(u) = -u\mathcal{G}(u), \tag{4.6b}$$

which is easily verified from (3.16a). Substituting (4.6b) into (4.5a), we obtain

$$\begin{aligned} C_0(\zeta,\tau) &= \zeta\mathcal{N}(u) + \sqrt{\alpha\tau}\,[\mathcal{G}(u')]_{-\infty}^{u} \\ &= \sqrt{\alpha\tau}[u\mathcal{N}(u) + \mathcal{G}(u)]. \end{aligned} \tag{4.7}$$

To calculate C_1 and C_2, we need the following derivatives:

$$\begin{aligned} \frac{\partial C_0}{\partial \tau} &= \frac{1}{2}\sqrt{\frac{\alpha}{\tau}}[u\mathcal{N}(u) + \mathcal{G}(u)] + \sqrt{\alpha\tau}[\mathcal{N}(u) + u\mathcal{G}(u) - u\mathcal{G}(u)]\frac{d}{dt}\left(\frac{\zeta}{\sqrt{\alpha\tau}}\right) \\ &= \frac{1}{2}\sqrt{\frac{\alpha}{\tau}}[u\mathcal{N}(u) + \mathcal{G}(u)] - \frac{u}{2\tau}\sqrt{\alpha\tau}\mathcal{N}(u) = \frac{1}{2}\sqrt{\frac{\alpha}{\tau}}\mathcal{G}(u) \\ &= \frac{\alpha\mathcal{K}(u)}{2}. \end{aligned} \tag{4.8}$$

Once we have the form in (4.8), then using (B.4), it is trivial to calculate the following derivatives:

$$\frac{\partial^2 C_0}{\partial\tau\partial\zeta} = \frac{\alpha}{2}\frac{\partial\mathcal{K}(u)}{\partial\zeta} = \frac{\alpha}{2}\left[-\frac{\mathcal{K}(u)}{\sqrt{2\alpha\tau}}H_1\left(\frac{u}{\sqrt{2}}\right)\right] = -\frac{\mathcal{G}(u)}{2\tau\sqrt{2}}H_1\left(\frac{u}{\sqrt{2}}\right), \tag{4.9}$$

$$\frac{\partial^2 C_0}{\partial\tau^2} = \frac{\alpha}{2}\frac{\partial\mathcal{K}(u)}{\partial\tau} = \frac{\alpha\mathcal{K}(u)}{8\tau}H_2\left(\frac{u}{\sqrt{2}}\right), \tag{4.10a}$$

$$\frac{\partial^3 C_0}{\partial\tau^3} = \frac{\alpha}{2}\frac{\partial^2\mathcal{K}(u)}{\partial\tau^2} = \frac{\alpha\mathcal{K}(u)}{32\tau^2}H_4\left(\frac{u}{\sqrt{2}}\right). \tag{4.10b}$$

Substituting (4.9) into (3.6), we obtain

$$\begin{aligned} C_1(\zeta,\tau) &= \gamma\tau^2\frac{\partial^2 C_0}{\partial\tau\partial\zeta} = -\frac{\gamma\tau\mathcal{G}(u)}{2\sqrt{2}}H_1\left(\frac{u}{\sqrt{2}}\right) = -\frac{\gamma\tau\mathcal{G}(u)}{2\sqrt{2}}\left(2\frac{u}{\sqrt{2}}\right) \\ &= -\frac{\gamma\tau u}{2}\mathcal{G}(u). \end{aligned} \tag{4.11}$$

Substituting (4.8) and (4.10) into (3.11), we obtain

$$C_2(\zeta,\tau) = \frac{\gamma^2}{\alpha}\tau^4\left[\frac{\alpha\mathcal{K}(u)}{32\tau^2}H_4\left(\frac{u}{\sqrt{2}}\right)\right] + Q_2\tau^3\left[\frac{\alpha\mathcal{K}(u)}{8\tau}H_2\left(\frac{u}{\sqrt{2}}\right)\right] + Q_1\tau^2\left[\frac{\alpha\mathcal{K}(u)}{2}\right]$$
$$= \frac{\alpha\tau^2}{2}\left[\frac{\gamma^2}{16\alpha}H_4\left(\frac{u}{\sqrt{2}}\right) + \frac{Q_2}{4}H_2\left(\frac{u}{\sqrt{2}}\right) + Q_1\right]\mathcal{K}(u). \tag{4.12}$$

Parameter Perturbations

To find the correct perturbation for the parameters, we use τ-derivatives in (3.13a):

$$\alpha_1(\zeta) = \frac{\alpha C_1}{\tau}\left(\frac{\partial C_0}{\partial\tau}\right)^{-1} = -\frac{\gamma\alpha u}{2}\mathcal{G}(u)\left[\frac{\alpha}{2}\frac{\mathcal{G}(u)}{\sqrt{\alpha\tau}}\right]^{-1} = -\gamma u\sqrt{\alpha\tau}$$
$$= -\gamma\zeta, \tag{4.13}$$

where we have used (4.8) and (4.11). The usefulness of this functional dependence will become apparent later. Note that with this dependence, however, we may need to be more careful about calculating hedges using the perturbed-parameter approach.

Equation (4.13) has a nice interpretation in terms of the "shadow costs". In particular, we would expect that the perturbation could be interpreted as

$$\alpha_{\text{new}} = \alpha + \sum_{j=1}^{n}\frac{\partial\alpha}{\partial x_j}\langle dx_j\rangle, \tag{4.14}$$

where $\langle dx_j\rangle$ is some average movement in x_j. But substituting (A.3a) into (4.13), we obtain

$$\sum_{j=1}^{n}\frac{\partial\alpha}{\partial x_j}\langle dx_j\rangle = -\epsilon\zeta\frac{1}{2\alpha}\sum_{j=1}^{n}x_j\frac{\partial\alpha}{\partial x_j}\beta_j$$
$$= \sum_{j=1}^{n}\left[-\frac{\epsilon\zeta}{2\alpha}x_j\beta_j\right]\frac{\partial\alpha}{\partial x_j}$$
$$\langle dx_j\rangle = -\frac{\epsilon\zeta}{2\alpha}x_j\beta_j, \tag{4.15}$$

where β_j is defined in Appendix A. Note that in the above, $\epsilon\zeta$ is just the deviation of the basket price from the strike price.

Continuing to next order, we use τ-derivatives in (3.13b):

$$\alpha_2(\zeta) = \left(C_2 - \frac{\alpha_1^2\tau^2}{2\alpha^2}\frac{\partial^2 C_0}{\partial\tau^2}\right)\left(\frac{\tau}{\alpha}\frac{\partial C_0}{\partial\tau}\right)^{-1}$$
$$= \left[\frac{\tau}{\alpha}\frac{\alpha\mathcal{K}(u)}{2}\right]^{-1}\left\{\frac{\alpha\tau^2}{2}\left[\frac{\gamma^2}{16\alpha}H_4\left(\frac{u}{\sqrt{2}}\right) + \frac{Q_2}{4}H_2\left(\frac{u}{\sqrt{2}}\right) + Q_1\right]\mathcal{K}(u)\right.$$

$$-\frac{\gamma^2\zeta^2\tau^2}{2\alpha^2}\frac{\alpha K(u)}{8\tau}H_2\left(\frac{u}{\sqrt{2}}\right)\Bigg\}$$
$$=\alpha\tau\left[\frac{\gamma^2}{16\alpha}H_4\left(\frac{u}{\sqrt{2}}\right)+\frac{Q_2}{4}H_2\left(\frac{u}{\sqrt{2}}\right)+Q_1\right]-\frac{\gamma^2\zeta^2\tau}{\alpha^2}\frac{\alpha}{8\tau}H_2\left(\frac{u}{\sqrt{2}}\right)$$
$$=\tau\left\{\frac{\gamma^2}{4}\left[\frac{1}{4}H_4\left(\frac{u}{\sqrt{2}}\right)-\left(\frac{u}{\sqrt{2}}\right)^2H_2\left(\frac{u}{\sqrt{2}}\right)\right]+\frac{\alpha Q_2}{4}H_2\left(\frac{u}{\sqrt{2}}\right)+\alpha Q_1\right\}. \tag{4.16}$$

Expanding the Hermite polynomials and combining terms, we obtain

$$\alpha_2(\zeta)=\alpha\tau\left[\frac{\gamma^2(-5u^2+3)}{4\alpha}+\frac{Q_2(u^2-1)}{2}+Q_1\right]. \tag{4.17}$$

Note that the u^4 terms in the bracketed expression in (4.16) canceled out. They must have canceled, or else our correction would have been $O(\epsilon^2\zeta^4)=O(\epsilon^{-2}\mathbf{x})$ as we exited the boundary layer.

Bullish Spread Options

Next we consider the *bullish spread* option, which has a payoff of the following form:

$$p_R(\mathbf{x})=\begin{cases}0, & S<K,\\ S-K, & K<S<K+a\epsilon,\\ a\epsilon, & S>K+a\epsilon,\end{cases} \tag{4.18}$$

where we denote the spread option by R. Rewriting the payoff function in the ζ-coordinates, we have

$$p_R(\zeta)=\begin{cases}0, & \zeta<0,\\ \epsilon\zeta, & 0<\zeta<a,\\ a\epsilon, & \zeta>a.\end{cases} \tag{4.19}$$

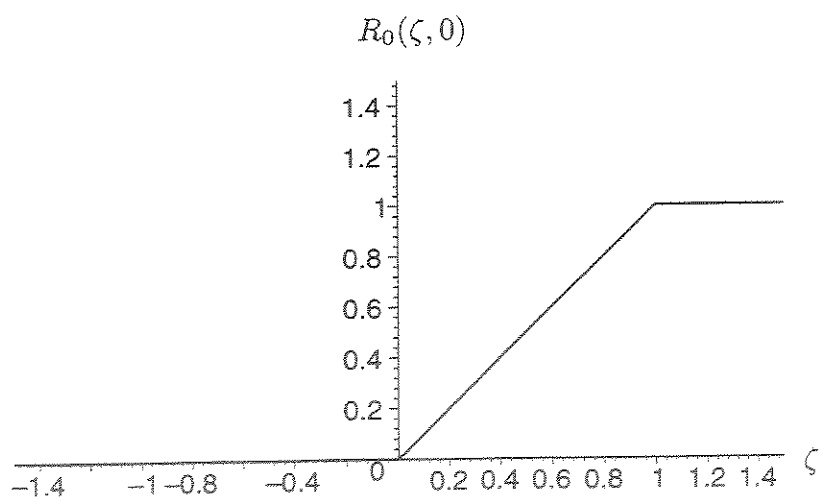

Figure 4.2. Bullish spread option payoff.

The graph is shown in Figure 4.2. However, we note from the graph that

$$p_R(\zeta) = p_C(\zeta) - p_C(\zeta - a). \tag{4.20}$$

Thus we have that

$$R(\zeta, \tau; \mathbf{x}) = C(\zeta, \tau; \mathbf{x}) - C(\zeta - a, \tau; \mathbf{x}), \tag{4.21}$$

and hence we may use (4.7), (4.11), and (4.12) to obtain a perturbation expansion for R. Equation (4.21) has a nice financial interpretation. In a spread option, you are betting that the market will go up, but not too far up. This saves money on the option. Equation (4.21) indicates that purchasing a spread is equivalent to purchasing a call option with strike price K, but then *selling* a call option with strike price $K + a\epsilon$. This second call is essentially a bet that the value of the basket will not rise above $K + a\epsilon$.

The parameter perturbation is slightly more subtle. Since our expressions for the perturbations α_1 and α_2 depend on ζ, we see that we must perturb each term in (4.21) individually. Thus we obtain

$$R(\zeta, \tau; \mathbf{x}) = C_0(\zeta, \tau; \alpha + \epsilon\alpha_1(\zeta) + \epsilon^2\alpha_2(\zeta)) - C_0(\zeta - a, \tau; \alpha + \epsilon\alpha_1(\zeta - a) + \epsilon^2\alpha_2(\zeta - a)). \tag{4.22}$$

Other Options

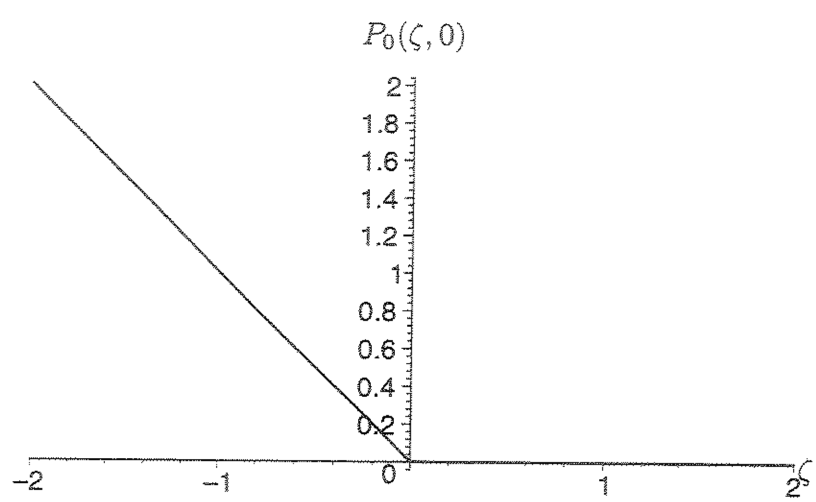

Figure 4.3. Put option payoff.

We conclude this section with a brief description of other options. A *put* option has a payoff which is the mirror image of the call option about $S = K$. Therefore, we have

$$p_P(\zeta) = [-\epsilon\zeta]^+ = \begin{cases} -\epsilon\zeta, & \zeta < 0, \\ 0, & \zeta > 0, \end{cases} \tag{4.23}$$

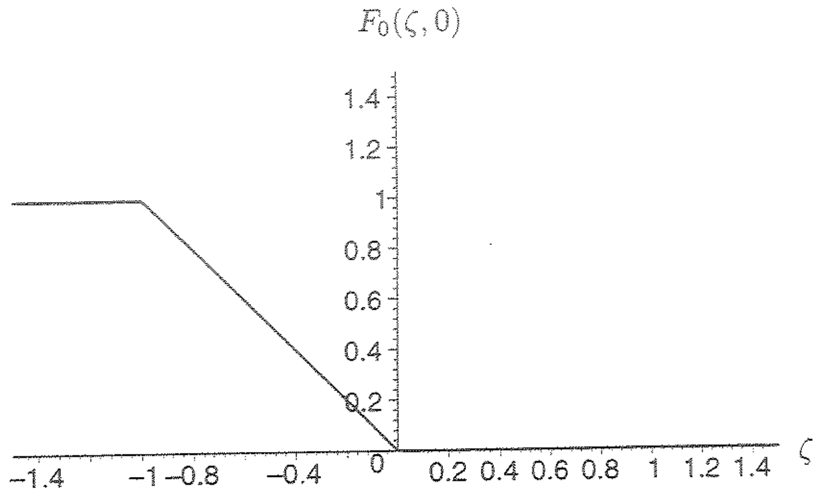

Figure 4.4. Bearish spread option payoff.

where we denote the put option by P. A graph of the payoff is shown in Figure 4.3. Since the governing equations have symmetry about the ζ-axis, we have

$$P(\zeta, \tau) = C(-\zeta, \tau). \tag{4.24}$$

Using (4.24), we can construct a perturbation expansion or perturbed-parameter expression for P.

Similarly, the *bearish spread* option has a payoff which is the mirror image of the bullish spread option about $S = K$. Therefore, we have

$$p_F(\zeta) = \begin{cases} a\epsilon, & \zeta < -a \\ -\epsilon\zeta, & -a < \zeta < 0, \\ 0, & \zeta > 0, \end{cases} \tag{4.25}$$

where we denote the bearish spread option by F. A graph of the payoff is shown in Figure 4.4. Hence

$$F(\zeta, \tau) = R(-\zeta, \tau). \tag{4.26}$$

Section 5: Digital Options

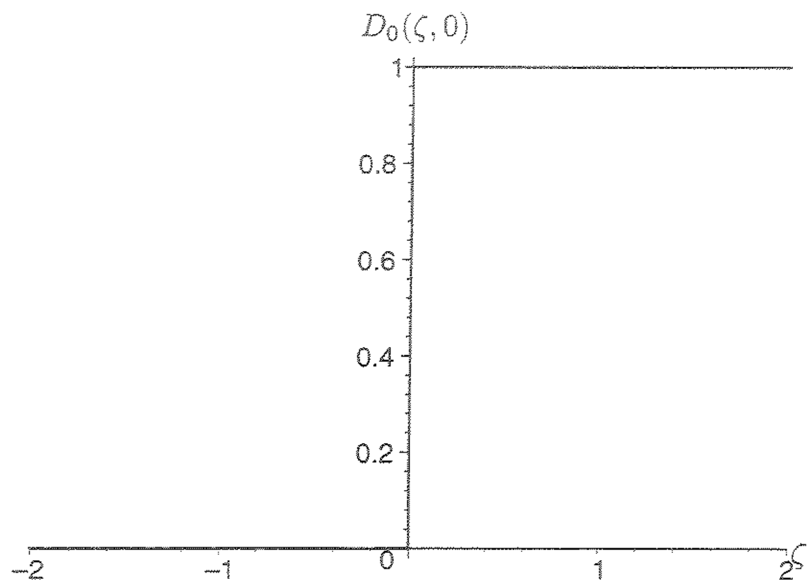

Figure 5.1. Digital call option payoff.

We next consider a *digital call* option, which pays only a fixed value. Here the payoff is given by

$$p_D(\mathbf{x}) = \begin{cases} 0, & S < K, \\ 1, & S > K, \end{cases} \tag{5.1}$$

which of course is a step function. Here we assume that the payoff value is 1. But since the problem is linear, it is a simple matter of scaling to obtain the case with general payoff. Therefore, we have a discontinuity in the function itself at $S = K$. Rewriting the payoff function in the ζ-coordinates, we have

$$p_D(\zeta) = \begin{cases} 0, & \zeta < 0, \\ 1, & \zeta > 0, \end{cases} \tag{5.2}$$

which is shown in Figure 5.1.

Due to the underlying linearity of our equations, we have that since

$$\epsilon p_D = \frac{dp_C}{d\zeta},$$

we must have that

$$\epsilon D(\zeta, \tau) = \frac{\partial C}{\partial \zeta}, \tag{5.3}$$

where D is the price of the digital option.

Using (5.3), we have that

$$D_n = \frac{\partial C_n}{\partial \zeta}. \tag{5.4}$$

Thus, from (4.7) we obtain

$$\begin{aligned} D_0(\zeta,\tau) &= \frac{\partial C_0}{\partial \zeta} = \sqrt{\alpha\tau}[\mathcal{N}(u) + u\mathcal{G}(u) - u\mathcal{G}(u)]\frac{d}{d\zeta}\left(\frac{\zeta}{\sqrt{\alpha\tau}}\right) \\ &= \mathcal{N}(u). \end{aligned} \tag{5.5}$$

In addition, we may use (B.4), (4.9), and (4.10) to obtain the needed expressions for D_1 and D_2:

$$\frac{\partial D_0}{\partial \tau} = \frac{\partial^2 C_0}{\partial \tau \partial \zeta} = -\frac{\mathcal{G}(u)}{2\tau\sqrt{2}} H_1\left(\frac{u}{\sqrt{2}}\right) = -\frac{\mathcal{G}(u)}{2\tau\sqrt{2}}\left(2\frac{u}{\sqrt{2}}\right) = -\frac{u\mathcal{G}(u)}{2\tau}. \tag{5.6a}$$

$$\frac{\partial^2 D_0}{\partial \tau^2} = \frac{\alpha}{2}\frac{\partial^2 \mathcal{K}(u)}{\partial \tau \partial \zeta} = -\frac{\alpha\mathcal{K}(u)}{8\tau\sqrt{2\alpha\tau}} H_3\left(\frac{u}{\sqrt{2}}\right), \tag{5.6b}$$

$$\frac{\partial^3 D_0}{\partial \tau^3} = \frac{\alpha}{2}\frac{\partial^3 \mathcal{K}(u)}{\partial \tau^2 \partial \zeta} = -\frac{\alpha\mathcal{K}(u)}{32\tau^2\sqrt{2\alpha\tau}} H_5\left(\frac{u}{\sqrt{2}}\right). \tag{5.6c}$$

$$\frac{\partial^2 D_0}{\partial \tau \partial \zeta} = \frac{\alpha}{2}\frac{\partial^2 \mathcal{K}(u)}{\partial \zeta^2} = \frac{\alpha\mathcal{K}(u)}{4\alpha\tau} H_2\left(\frac{u}{\sqrt{2}}\right) = \frac{\mathcal{K}(u)}{4\tau} H_2\left(\frac{u}{\sqrt{2}}\right), \tag{5.7}$$

Substituting (5.7) into (3.6), we obtain

$$\begin{aligned} D_1(\zeta,\tau) &= \gamma\tau^2\frac{\partial^2 D_0}{\partial \tau \partial \zeta} = \frac{\gamma\tau}{4}\left(\frac{\mathcal{G}(u)}{\sqrt{\alpha\tau}}\right) H_2\left(\frac{u}{\sqrt{2}}\right) \\ &= \frac{\gamma}{4}\sqrt{\frac{\tau}{\alpha}}\left[4\left(\frac{u}{\sqrt{2}}\right)^2 - 2\right]\mathcal{G}(u) = \frac{\gamma\mathcal{G}(u)}{2}\sqrt{\frac{\tau}{\alpha}}(u^2-1). \end{aligned} \tag{5.8}$$

Substituting (5.6) into (3.11), we obtain

$$\begin{aligned} D_2(\zeta,\tau) &= \frac{\gamma^2}{\alpha}\tau^4\left[-\frac{\alpha\mathcal{K}(u)}{32\tau^2\sqrt{2\alpha\tau}} H_5\left(\frac{u}{\sqrt{2}}\right)\right] + Q_2\tau^3\left[-\frac{\alpha\mathcal{K}(u)}{8\tau\sqrt{2\alpha\tau}} H_3\left(\frac{u}{\sqrt{2}}\right)\right] \\ &\quad + Q_1\tau^2\left[-\mathcal{K}(u)\sqrt{\alpha\tau}\frac{u}{2\tau}\right] \\ &= -\frac{\tau^{3/2}}{2}\alpha\left[\frac{\gamma^2}{16\alpha\sqrt{2}} H_5\left(\frac{u}{\sqrt{2}}\right) + \frac{Q_2}{4\sqrt{2}} H_3\left(\frac{u}{\sqrt{2}}\right) + Q_1 u\right]\mathcal{K}(u). \end{aligned}$$

For later purposes, it will be more convenient to write this term as a single polynomial. Expanding the Hermite functions, we obtain

$$D_2(\zeta,\tau) = \frac{u\tau}{8}\left(Q_4 + Q_5 u^2 - \frac{\gamma^2 u^4}{\alpha}\right)\mathcal{G}(u), \tag{5.9}$$

$$Q_4 = -15\frac{\gamma^2}{\alpha} + 6Q_2 - 4Q_1,$$

$$Q_4 = 10\frac{\gamma^2}{\alpha} - 2Q_2.$$

Parameter Perturbations

Now we attempt to do the same sort of parameter perturbation as in section 4. Using (5.6a) in (3.13a), we obtain

$$\alpha_1(\zeta) = \frac{\alpha D_1}{\tau}\left(\frac{\partial D_0}{\partial \tau}\right)^{-1} = \frac{\gamma \mathcal{G}(u)}{2}\sqrt{\frac{\alpha}{\tau}}(u^2-1)\left[-\frac{u\mathcal{G}(u)}{2\tau}\right]^{-1}$$
$$= \frac{\gamma\sqrt{\alpha\tau}}{u}(1-u^2). \qquad (5.10)$$

But now we have a problem: α_1 vanishes when $\zeta = 0$. Why does this happen? From (5.6a), we see that

$$\frac{\partial D_0}{\partial \alpha}(0,\tau) = \frac{\tau}{\alpha}\frac{\partial D_0}{\partial \tau}(0,\tau) = 0.$$

Hence we are dividing by zero. This occurs because the digital call is symmetric about $\zeta = 0$, always fixed at the value $D_0(0,\tau) = 1/2$.

Thus, we must be more clever. We replace (3.12) by the following:

$$D_0(\zeta,\tau;\alpha) + \epsilon D_1(\zeta,\tau;\alpha) + \epsilon^2 D_2(\zeta,\tau;\alpha) = D_0(\zeta + \epsilon\zeta_1, \tau; \alpha + \epsilon\alpha_1 + \epsilon^2\alpha_2). \qquad (5.11)$$

We assume that $\zeta_2 = 0$, and verify this below. Expanding (5.11), we obtain

$$D_0 + \epsilon D_1 + \epsilon^2 D_2 = D_0 + \epsilon\alpha_1\frac{\partial D_0}{\partial \alpha} + \epsilon\zeta_1\frac{\partial D_0}{\partial \zeta} + \epsilon^2\alpha_2\frac{\partial D_0}{\partial \alpha} + \epsilon^2\alpha_1\zeta_1\frac{\partial^2 D_0}{\partial\alpha\partial\zeta}$$
$$+ \frac{\epsilon^2\alpha_1^2}{2}\frac{\partial^2 D_0}{\partial \alpha^2} + \frac{\epsilon^2\zeta_1^2}{2}\frac{\partial^2 D_0}{\partial \zeta^2}$$
$$D_1 = \frac{\alpha_1\tau}{\alpha}\frac{\partial D_0}{\partial \tau} + \zeta_1\frac{\partial D_0}{\partial \zeta}, \qquad (5.12a)$$
$$D_2 = \left(\frac{\alpha_2\tau}{\alpha} + \frac{\zeta_1^2}{2}\frac{2}{\alpha}\right)\frac{\partial D_0}{\partial \tau} + \frac{\alpha_1\zeta_1\tau}{\alpha}\frac{\partial^2 D_0}{\partial\tau\partial\zeta} + \frac{\alpha_1^2\tau^2}{2\alpha^2}\frac{\partial^2 D_0}{\partial \tau^2}. \qquad (5.12b)$$

Thus we will need the following term:

$$\frac{\partial D_0}{\partial \zeta} = \mathcal{G}(u)\frac{du}{d\zeta} = \frac{1}{\sqrt{\alpha\tau}}\mathcal{G}(u). \qquad (5.13)$$

We begin by solving (5.12a). To do so, we use (5.6a) and (5.13):

$$-\frac{\gamma}{2}\sqrt{\frac{\tau}{\alpha}}(1-u^2)\mathcal{G}(u) = \alpha_1\left[-\frac{u}{2\alpha}\mathcal{G}(u)\right] + \zeta_1\left[\frac{1}{\sqrt{\alpha\tau}}\mathcal{G}(u)\right]$$
$$-\frac{\gamma}{2}\sqrt{\frac{\tau}{\alpha}}(1-u^2) = -\frac{u\alpha_1}{2\alpha} + \frac{\zeta_1}{\sqrt{\alpha\tau}}.$$

Previously, the divergent terms occurred because we divided the constant term (in u) on the left by the u on the right. Now we avoid this problem by letting the ζ_1 term match the constant term:

$$
\begin{aligned}
-\frac{\gamma}{2}\sqrt{\frac{\tau}{\alpha}} &= \frac{\zeta_1}{\sqrt{\alpha\tau}} \\
\zeta_1 &= -\frac{\gamma\tau}{2},
\end{aligned}
\tag{5.14a}
$$

which implies that

$$
\begin{aligned}
\frac{\gamma u^2}{2}\sqrt{\frac{\tau}{\alpha}} &= -\frac{u\alpha_1}{2\alpha} \\
\alpha_1 &= -u\gamma\sqrt{\alpha\tau} = -\gamma\zeta.
\end{aligned}
\tag{5.14b}
$$

Note that (in this formulation) the α_1 here agrees with (4.13). This similarity is what led us to guess that the general case could always be done and try the analysis in section 2.

The ζ_1 term implies a drift of origin due to the fact that we are approximating a basket of stocks as a single stock.

Next we solve (5.12b), using (5.6b), (5.7), (5.8), and (5.14):

$$
\begin{aligned}
D_2 &= \left(\frac{\alpha_2\tau}{\alpha} + \frac{\gamma^2\tau^2}{4\alpha}\right)\left[-\frac{u\mathcal{G}(u)}{2\tau}\right] + \frac{\gamma\tau\zeta}{2\alpha}\left(\gamma\tau^2\frac{\partial^2 D_0}{\partial\tau\partial\zeta}\right) + \frac{\gamma^2\zeta^2\tau^2}{2\alpha^2}\left[-\frac{\alpha\mathcal{K}(u)}{8\tau\sqrt{2\alpha\tau}}H_3\left(\frac{u}{\sqrt{2}}\right)\right] \\
2D_2 &= -\left(\alpha_2 + \frac{\gamma^2\tau}{4}\right)\frac{u\mathcal{G}(u)}{\alpha} + \gamma u\sqrt{\frac{\tau}{\alpha}}D_1 - \frac{\gamma^2u^2\tau^2}{8\sqrt{2\alpha\tau}}\frac{\mathcal{G}(u)}{\sqrt{\alpha\tau}}H_3\left(\frac{u}{\sqrt{2}}\right) \\
\frac{2D_2}{u} &= -\left(\alpha_2 + \frac{\gamma^2\tau}{4}\right)\frac{\mathcal{G}(u)}{\alpha} + \gamma\sqrt{\frac{\tau}{\alpha}}\left[\frac{\gamma\mathcal{G}(u)}{2}\sqrt{\frac{\tau}{\alpha}}(u^2-1)\right] \\
&\quad - \frac{\gamma^2u\tau}{8\alpha\sqrt{2}}\mathcal{G}(u)\left[8\left(\frac{u}{\sqrt{2}}\right)^3 - 12\left(\frac{u}{\sqrt{2}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{2\alpha D_2}{u\mathcal{G}(u)} &= -\left(\alpha_2 + \frac{\gamma^2\tau}{4}\right) + \frac{\gamma^2\tau}{2}(u^2-1) - \frac{\gamma^2u^2\tau}{4}(u^2-3) \\
\alpha_2 &= \frac{\gamma^2\tau}{4}(5u^2-3) - \frac{\gamma^2u^4\tau}{4} - \frac{2\alpha}{u\mathcal{G}(u)}\left[\frac{u\tau}{8}\left(Q_4 + Q_5u^2 - \frac{\gamma^2u^4}{\alpha}\right)\mathcal{G}(u)\right] \\
&= \frac{\tau}{2}\left[\frac{\gamma^2}{2}(5u^2-3) - \frac{\alpha}{2}(Q_4 + Q_5u^2)\right].
\end{aligned}
\tag{5.15}
$$

Note that in this case we essentially had two degrees of freedom, as we could have introduced ζ_2. However, since we can obtain a nonsingular parameter perturbation with $\zeta_2 = 0$, we do so.

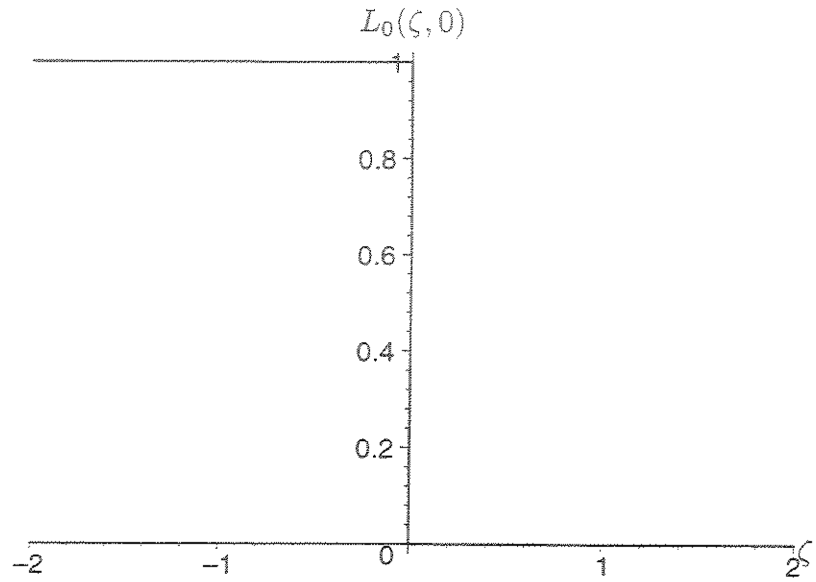

Figure 5.2. Digital put option payoff.

Other Options

As in the previous section, we can relate the *digital put* option to the digital call by just reflecting about the line $S = K$. Therefore, we have

$$p_L(\zeta) = \begin{cases} 1, & \zeta < 0, \\ 0, & \zeta > 0, \end{cases} \tag{5.16}$$

where we denote the digital put option by L. A graph of the payoff is shown in Figure 5.2. Since the governing equations have symmetry about the ζ-axis, we have

$$L(\zeta, \tau) = D(-\zeta, \tau). \tag{5.17}$$

Using (5.17), we may construct a perturbation expansion or perturbed-parameter expression for L.

Section 6: Barrier Model Formulation

We next consider a simple model for a *"down-and-out" barrier call.* In such a system, if S falls below a certain value $K - y\epsilon$ for *any* τ, then the option becomes worthless. Therefore, we have the following boundary condition:

$$B(S = K - y\epsilon, \tau) = 0, \tag{6.1a}$$

where the B stands for barrier option. Note that the domain for the problem is now

$$S > K - y\epsilon \qquad \Longrightarrow \qquad \sum_{j=1}^{n} e^{-\mu_j \tau} x_j > K - y\epsilon. \tag{6.1b}$$

This type of option is cheaper than a standard call option because if the asset price falls below $K - y\epsilon$, then rises again, the option still cannot be exercised. However, for the option to be significantly cheaper, the barrier must be close (at least in volatility terms) to the strike price, since wide swings in asset prices are unlikely.

We note from (6.1b) that if we change to the ζ-coordinate system, we will have a boundary that moves in time. Therefore, we make the assumption that μ_k is small. This is a reasonable assumption, since interest rates vary slowly. We let $\mu_j = \mu_{0j}\phi(\epsilon)$, where $\phi(\epsilon) = o(1)$. Thus (6.1b) becomes, to leading orders,

$$\sum_{j=1}^{n} x_j - K > -y\epsilon + \sum_{j=1}^{n} \mu_{0j} x_j \phi(\epsilon)\tau$$

$$\zeta > -y + \frac{\phi(\epsilon)\tau}{\epsilon}\mu, \qquad \mu = \sum_{j=1}^{n} \mu_{0j} x_j. \tag{6.2}$$

We note from (6.2) that we have several cases. If $\phi(\epsilon) > O(\epsilon)$, then the slowly-varying limit is not appropriate. If $\phi(\epsilon) = \epsilon$, then we still have an undesirable time-dependent boundary. Therefore, we desire $\phi(\epsilon) = o(\epsilon)$, and for simplicity we take $\phi(\epsilon) = \epsilon^2$, so the domain is given by

$$\zeta > -y + \mu\epsilon\tau,$$

and (6.1a) becomes

$$B(\zeta = -y + \mu\epsilon\tau, \tau) = 0. \tag{6.3}$$

Since we will work in ζ-space from now on, we will drop the ζ inside the boundary condition.

Here the payoff is given by

$$p_B(\mathbf{x}) = [S - K]^+ = \begin{cases} 0, & K - y < S < K, \\ S - K, & S > K, \end{cases}$$

which becomes

$$p_B(\zeta) = [\epsilon\zeta]^+ = \begin{cases} 0, & -y < \zeta < 0, \\ \epsilon\zeta, & \zeta > 0, \end{cases} \tag{6.4}$$

and again we have

$$V_{n+1}(\zeta,\tau) = B_n(\zeta,\tau), \quad n > 0, \tag{6.5}$$

where the B denotes the barrier option. In addition, we now have the following boundary conditions:

$$B(-y+\mu\epsilon\tau,\tau) = 0$$

$$B_0(-y,\tau) + \mu\epsilon\tau\frac{\partial B_0}{\partial\zeta}(-y,\tau) + B_1(-y,\tau) = 0$$

$$B_0(-y,\tau) = 0, \tag{6.6a}$$

$$B_1(-y,\tau) = -\mu\tau\frac{\partial B_0}{\partial\zeta}(-y,\tau). \tag{6.6b}$$

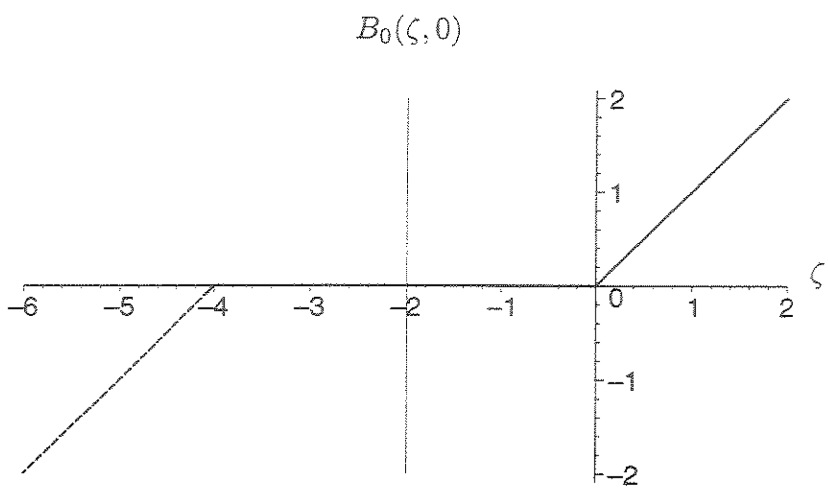

Figure 6.1. Barrier option payoff (solid line) and extension (dotted line).

The payoff is given by the solid line in Figure 6.1. We note that the payoff is the same as the call, except that we have the zero boundary condition (6.6a) on B_0. Therefore, we may extend the problem in an odd way about $\zeta = -y$, as shown by the dotted line in Figure 6.1. Thus, by the method of images we have that

$$B_0(\zeta,\tau) = C_0(\zeta,\tau) - C_0(-(\zeta+2y),\tau). \tag{6.7}$$

However, this trick will not work at the next order due to the boundary condition (6.6b). Substituting (6.7) into (6.6b) and using (5.5), we obtain

$$B_1(-y,\tau) = -\mu\tau\left[\frac{\partial C_0}{\partial\zeta}(-y,\tau) + \frac{\partial C_0}{\partial\zeta}(-y,\tau)\right]$$

$$= -2\mu\tau\mathcal{N}\left(-\frac{y}{\sqrt{\alpha\tau}}\right). \tag{6.8}$$

The solution for B_1 thus involves the following steps. First, we substitute our solution for B_0 into (3.6) in order to find a particular solution for B_1. Then if this solution does not satisfy (6.8), we must introduce a homogeneous solution to the problem. This is the example to which we alluded after (3.7).

Section 7: Numerical Simulations

We now compare our asymptotic results to numerical simulations of (2.23). The boundary conditions used were for a standard call option. A finite-difference scheme was used, second-order in space, and first-order backward Euler in time, with an implicit scheme for time advancing. At each new time level, the boundary conditions on the edge of the domain are obtained from an explicit time-stepping scheme.

The numerical simulations were run in various dimensions, with $\epsilon = 0.1$ and $z_{jk} = \delta_{jk}$.

We begin by examining the one-dimensional case. Here the spatial domain is taken to be $x \in [0, 1]$, with the strike price $K = 5/9$.

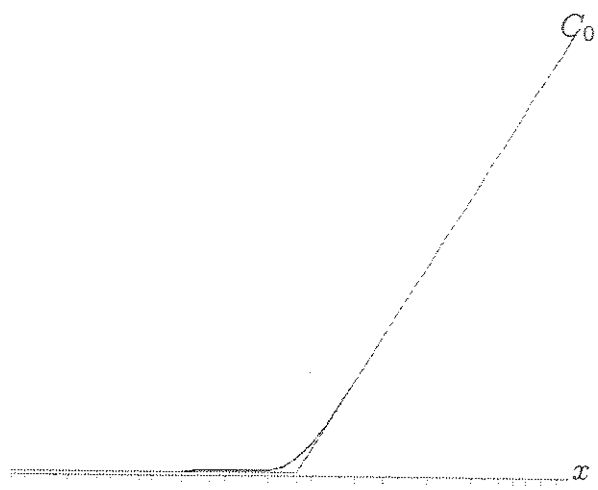

Figure 7.1. C_0 *vs.* x. Purple line: $t = 0$. Red line: $t = 1/2$.

First, we plot the first three terms in our asymptotic expansion for C. Figure 7.1 shows a graph of C_0 *vs.* x for $t = 0$ and $t = 1/2$. Figure 7.2 shows a graph of C_1 *vs.* x for $t = 1/2$ and $t = 2$. The y-axis here is scaled by a factor of 300. Figure 7.3 shows a graph of C_1 *vs.* x for $t = 1/2$ and $t = 2$. The y-axis here is scaled by a factor of 1000.

Next we compare the numerical and asymptotic schemes. The time step is set at $\Delta t = 0.02$ and four separate runs were performed with different values of Δx. We denote the calculation with $\Delta x = 1/2^{i-1}9$ as "level i."

Figures 7.4–7.6 show the convergence of the finite-difference solution to the first three orders of the perturbation solution as Δx decreases. Here $t = 1/2$ and the plotting x-range has been truncated to $[0.4, 0.7]$. In each graph, the top curve is the perturbation solution.

For completeness, we present some multidimensional numerical results. In two dimensions, the domain was taken to be $[0, 1] \times [0, 1]$, the weights were taken to be $w_1 = 1$, $w_2 = 1$, and the strike price was taken to be $K = 5/9$. Figure 7.7 shows a graph of the

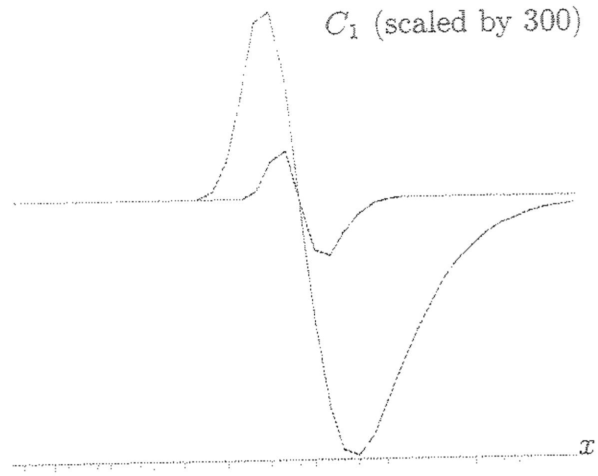

Figure 7.2. C_1 *vs.* x. Red line: $t = 1/2$. Purple line: $t = 2$.

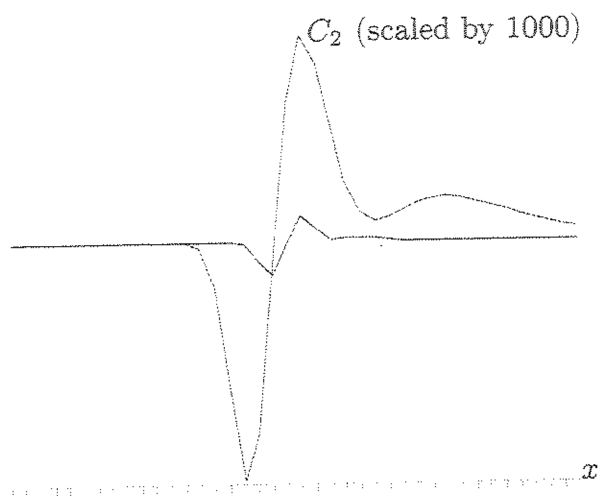

Figure 7.3. C_2 *vs.* x. Red line: $t = 1/2$. Purple line: $t = 2$.

solution for $t = 1/2$ with $\Delta t = 0.02$ and $\Delta x = 1/9$. The graph is shown in red, the grid is shown in green, and the initial condition (barely visible near the strike price) is shown in blue.

In three dimensions, the domain was taken to be $[0, 1] \times [0, 1] \times [0, 1]$, the weights were taken to be $w_1 = 1$, $w_2 = 1$, $w_3 = 1$, and the strike price was taken to be $K = 5/9$. Again we take $t = 1/2$ with $\Delta t = 0.02$ and $\Delta x = 1/9$. Since the full three-dimensional solution cannot be displayed, Figures 7.8 and 7.9 show the solution for fixed values of x_3. Again, the graph is shown in red, the grid is shown in green, and the initial condition (barely visible near the strike price) is shown in blue.

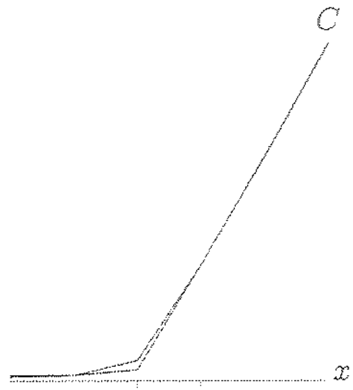

Figure 7.4. Asymptotic (purple) and computational (red) solution, $t = 1/2$, $\Delta x = 1/9$.

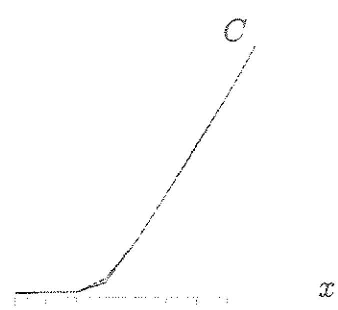

Figure 7.5. Asymptotic (purple) and computational (red) solution, $t = 1/2$, $\Delta x = 1/18$.

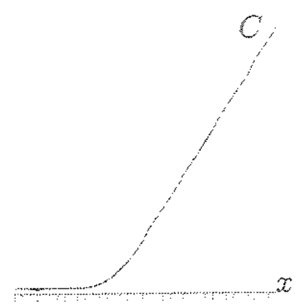

Figure 7.6. Asymptotic (purple) and computational (red) solution, $t = 1/2$, $\Delta x = 1/72$.

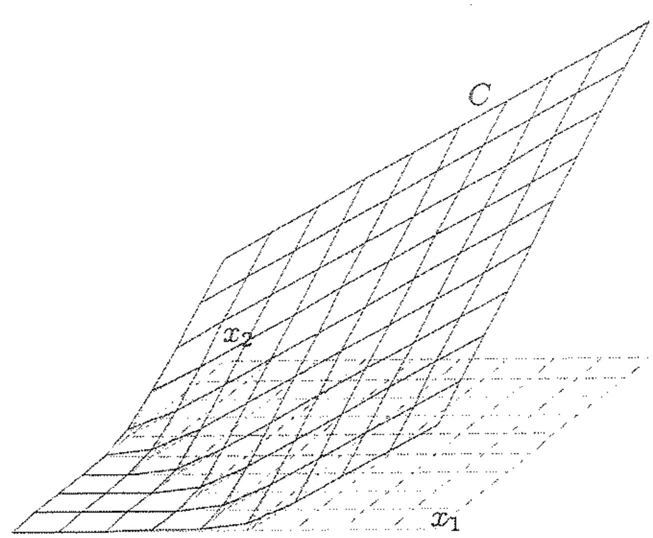

Figure 7.7. Two-dimensional numerical solution, $t = 1/2$.

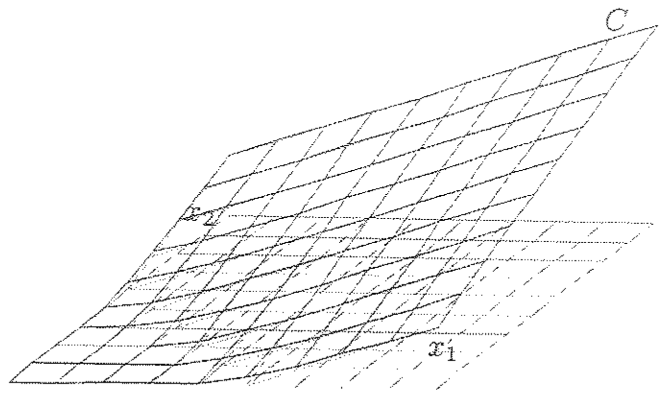

Figure 7.8. Slice of three-dimensional numerical solution, $t = 1/2$, $x_3 = 1/9$.

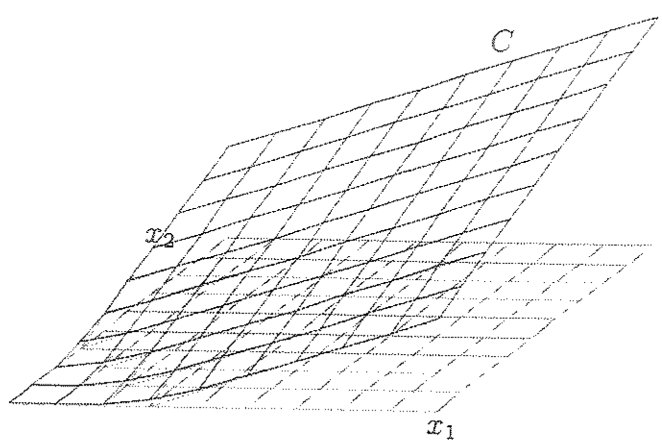

Figure 7.9. Slice of three-dimensional numerical solution, $t = 1/2$, $x_3 = 0$.

Section 8: Conclusions and Further Research

A fast accurate method was developed to calculate approximate values of some common options on a basket, based on a general perturbation procedure and renormalization, under the assumption of small volatility of the asset prices. The form of the payoff function is relevant to the applicability of this method, and the case of Asian options, for example, cannot be treated without some modification. The results from some simple numerical calculations not only show the accuracy of the asymptotic method, but also how unwieldy calculations of the full system become as the number of assets in the basket increases. Future research may consider the case of Asian options, and American call and put options.

Section 9: Nomenclature

In the manuscript, boldface or the arrow notation indicates a vector where the componenets are the italic letters with subscript k. The equation number where a particular quantity first appears is listed, if appropriate.

A_k: foreign exchange rate for currency k (2.2).
a: top of payoff range for spread option (4.18).
b_k: volatility for foreign exchange A_k (2.2).
$B(\cdot, \tau)$: value of barrier option (6.1a).
$C(\zeta, \tau)$: value of call option (4.3).
$D(\zeta, \tau)$: value of digital option (5.3).
dW_k: Wiener process for asset price S_k (2.1).
dZ_k: Wiener process for foreign exchange rate A_k (2.2).
$F(\zeta, \tau)$: value of bearish spread option (4.26).
f: arbitrary function, variously defined.
$\mathcal{G}(\cdot)$: Gaussian (normal) probability density function (3.15b).
$H(\cdot)$: Hermite polynomial (3.17a).
i: indexing variable.
j: indexing variable.
$\mathcal{K}(\cdot)$: diffusion kernel (3.15a).
K: strike price for option.
k: indexing variable.
$\mathcal{L}$: diffusion operator (3.2a).
$L(\zeta, \tau)$: value of digital put option (5.17).
M: matrix whose jkth entry is z_{jk} (A.1a).
$\mathcal{N}(u)$: Gaussian probability mass (cumulative normal) function (4.5a).
n: number of options in basket (2.5).
$P(\zeta, \tau)$: value of put option (4.24).
$p(\cdot)$: payoff function for option (2.24b).
Q: constant, variously defined.
q_k: units of currency k in portfolio (2.5).
$R(\zeta, \tau)$: value of spread option (4.21).
r: risk-free rate of return (2.7b).
S: asset or basket price (2.1).
s: Laplace transform variable (B.1).
t: time from option sale (2.1).
u: similarity variable for diffusion equation (3.15a).
$\tilde{V}(\mathbf{x}, \tau)$: outer solution for discounted option price (2.18).
$V(\zeta, \tau)$: inner solution for discounted option price (2.22).
$\tilde{v}(\mathbf{S}, t)$: value of option at time t (2.5).

$v(\mathbf{S}, \tau)$: discounted value of option at time τ (2.15).
w_k: weight of asset k in basket (2.15).
x_k: weighted future price of stock k (2.18).
y: negative of the barrier value in the ζ-coordinate system (6.1).
$\mathcal{Z}$: the integers.
z_{jk}: scaled correlation parameter (2.20).
α: parameter, value $\mathbf{x}^T M \mathbf{x}$ (3.2a).
β_j: parameter, value $(M\mathbf{x})_j$ (4.15).
γ: parameter (3.4a).
δ_{jk}: Kronecker delta function.
ϵ: small dimensionless parameter (2.20).
ζ: scaled variable about strike price (2.22).
λ_k: units of asset k in portfolio (2.5).
$\tilde{\mu}_k$: unknown drift rate for asset k (2.1).
μ_k: known adapted "drift rate" for martingale distribution, value $r_k - \sigma_k b_k \rho_k$ (2.14).
ν_k: drift rate for currency rate A_k (2.2).
$\pi(\mathbf{S}, t)$: portfolio value (2.5).
ρ: correlation parameter (2.3).
σ_k: volatility for asset k (2.1).
τ: time variable measured backwards from the exercise date (2.15).

Other Notation

B: as a subscript on p, used to indicate a barrier option (6.4).
C: as a subscript on p, used to indicate a call option (4.1).
D: as a subscript on p, used to indicate a digital option (5.1).
F: as a subscript on p, used to indicate a bearish spread option (4.25).
ex: as a subscript, used to indicate the exercise date (2.12).
L: as a subscript on p, used to indicate a bearish spread option (5.16).
new: as a subscript on α, used to indicate a perturbed parameter (4.14).
P: as a subscript on p, used to indicate a put option (4.23).
p: as a subscript, used to indicate a particular solution (3.5).
R: as a subscript on p, used to indicate a bullish spread option (4.18).
$\hat{}$: used to indicate the Laplace transform (B.1).
$'$: used to indicate a dummy variable (3.15a).
$[\cdot]^+$: $\max\{\cdot, 0\}$ (4.1).
$n \in \mathcal{Z}$: as a subscript, used to index over assets (2.1), currency rates (2.2), to indicate an expansion in ϵ (3.1), or simply to keep track of different parameters (3.9).

Appendix A: Parameter Simplifications

We begin by simplifying α. We note that if we define the $\mathcal{R}^{n\times n}$ correlation matrix M by

$$M = (z_{jk}), \tag{A.1a}$$

then (3.2b) may be written in the more compact notation

$$\alpha = \mathbf{x}^T M \mathbf{x}. \tag{A.1b}$$

Here M is positive definite. Unfortunately, this seems as far as the vector notation will get us.

Next we simplify γ. We begin by deriving an expression for the partial derivative:

$$\frac{\partial \alpha}{\partial x_j} = \sum_{i=1}^{n}\sum_{k=1}^{n} \rho_{ik}\frac{\partial(x_i x_k)}{\partial x_j} = \sum_{i=1}^{n}\sum_{k=1}^{n} \rho_{ik}(x_i\delta_{jk} + x_k\delta_{ij}) = 2\sum_{k=1}^{n} z_{jk}x_k,$$

where δ_{ik} is the Kronecker delta and in the last equation we have used the symmetry of M. The last sum comes up often, so we define it as a new quantity:

$$\beta_j = \sum_{k=1}^{n} z_{jk}x_k \qquad \Longrightarrow \qquad \vec{\beta} = M\mathbf{x}, \tag{A.2a}$$

$$\frac{\partial \alpha}{\partial x_j} = 2\beta_j. \tag{A.2b}$$

Since we are summing over both j and k, each of the partial derivatives in (3.4b) gets counted twice, once as a particular j value and once as a particular k value. Thus, we may rewrite (3.4b) as

$$\gamma = \frac{1}{2\alpha}\sum_{j=1}^{n} x_j \frac{\partial \alpha}{\partial x_j}\sum_{k=1}^{n} z_{jk}x_k = \frac{1}{2\alpha}\sum_{j=1}^{n} x_j \frac{\partial \alpha}{\partial x_j}\beta_j \tag{A.3a}$$

$$= \frac{1}{\alpha}\sum_{j=1}^{n} x_j\beta_j\beta_j = \frac{1}{\alpha}\sum_{j=1}^{n} x_j\beta_j^2, \tag{A.3b}$$

where we have used (A.2). Unfortunately, there doesn't seem to be an easy way to rewrite this in vector notation.

Next we compute Q_1, first computing an intermediate expression:

$$\frac{\partial^2 \alpha}{\partial x_j x_k} = 2\sum_{i=1}^{n} \rho_{ji}\frac{\partial x_i}{\partial x_k} = 2\sum_{i=1}^{n} \rho_{ji}\delta_{ik} = 2z_{jk}$$

$$Q_1 = \frac{1}{2\alpha}\sum_{j=1}^{n} x_j \sum_{k=1}^{n} z_{jk}^2 x_k, \tag{A.4}$$

where we have used (A.2).

In order to compute Q_2, we need the following derivative:

$$\begin{aligned}\frac{\partial\gamma}{\partial x_j} &= -\frac{1}{\alpha^2}\frac{\partial\alpha}{\partial x_j}\sum_{k=1}^{n} x_k\beta_k^2 + \frac{1}{\alpha}\sum_{k=1}^{n}\frac{\partial x_k}{\partial x_j}\beta_k^2 + \frac{2}{\alpha}\sum_{k=1}^{n} x_k\beta_k\frac{\partial\beta_k}{\partial x_j}\\ &= -\frac{2\beta_j}{\alpha}\left(\frac{1}{\alpha}\sum_{k=1}^{n} x_k\beta_k^2\right) + \frac{1}{\alpha}\sum_{k=1}^{n}\delta_{kj}\beta_k^2 + \frac{2}{\alpha}\sum_{k=1}^{n} x_k\beta_k z_{jk}\\ &= -\frac{2\beta_j\gamma}{\alpha} + \frac{\beta_j^2}{\alpha} + \frac{2}{\alpha}\sum_{k=1}^{n} x_k\beta_k z_{jk},\end{aligned} \tag{A.5}$$

where we have used (A.2) and (A.3).

Substituting (A.2b) and (A.5) into (3.9b) and using the trick about summing twice, we obtain

$$\begin{aligned}Q_2 &= \frac{1}{6\alpha}\sum_{j=1}^{n} x_j\sum_{k=1}^{n} z_{jk}x_k\left[\frac{(2\beta_j)(2\beta_k)}{\alpha} + 4\frac{\partial\gamma}{\partial x_j}\right]\\ &= \frac{2}{3\alpha^2}\sum_{j=1}^{n} x_j\sum_{k=1}^{n} z_{jk}x_k\left[\beta_j\beta_k - 2\beta_j\gamma + \beta_j^2 + 2\sum_{i=1}^{n} x_i\beta_i\rho_{ij}\right]\\ &= \frac{2}{3\alpha^2}\sum_{j=1}^{n} x_j\left\{\beta_j\left[-2\gamma\beta_j + \beta_j^2 + 2\sum_{i=1}^{n} x_i\beta_i\rho_{ij}\right] + \sum_{k=1}^{n} z_{jk}x_k\beta_j\beta_k\right\}\\ &= -\frac{4\gamma^2}{3\alpha} + \frac{2}{3\alpha^2}\sum_{j=1}^{n} x_j\left\{\beta_j^3 + 3\sum_{k=1}^{n} z_{jk}x_k\beta_j\beta_k\right\}\\ &= -\frac{4\gamma^2}{3\alpha} + \frac{2}{3\alpha^2}\sum_{j=1}^{n} x_j\beta_j\left\{\beta_j^2 + 3\sum_{k=1}^{n} z_{jk}x_k\beta_k\right\}.\end{aligned} \tag{A.6}$$

Appendix B: General Differentiation Formulas

We would like to calculate systematically the necessary derivatives to obtain V_1 and V_2. To do so, we introduce the standard concept of a Laplace transform:

$$\hat{f}(\zeta, s) = \int_0^\infty e^{-s\tau} f(\zeta, \tau)\, d\tau, \qquad f(\zeta, \tau) = \int_C e^{s\tau} \hat{f}(\zeta, s)\, ds, \tag{B.1}$$

where C is the Bromwich contour. A useful transform pair for our work may be found in [5], 29.3.87:

$$\begin{aligned} s^{(j-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right) &\iff \frac{1}{2^j\sqrt{\pi\tau^{j+1}}} \exp\left(-\frac{\zeta^2}{2\alpha\tau}\right) H_j\left(\frac{\zeta}{\sqrt{2\alpha\tau}}\right) \\ \left(\frac{2s}{\alpha}\right)^{(j-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right) &\iff \frac{2^{(j-1)/2}}{2^{j-1/2}\alpha^{j/2-1}\tau^{j/2}} \frac{\mathcal{G}(u)}{\sqrt{\alpha\tau}} H_j\left(\frac{u}{\sqrt{2}}\right) \\ \left(\frac{2s}{\alpha}\right)^{(j-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right) &\iff \frac{\alpha}{(2\alpha\tau)^{j/2}} \mathcal{K}(u) H_j\left(\frac{u}{\sqrt{2}}\right). \end{aligned} \tag{B.2}$$

We note that in (3.15a) there is no dependence of V_0 on ζ and τ through p. Therefore, we need only calculate the derivatives of $\mathcal{K}$, which we do using Laplace transforms. Using the transform pair (B.2) with $n = 0$, we have that

$$\begin{aligned} \hat{\mathcal{K}}(u) &= \frac{1}{\alpha}\left(\frac{2s}{\alpha}\right)^{(0-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right) \\ \frac{\partial^j \hat{\mathcal{K}}}{\partial \tau^j}(u) &= \frac{s^j}{\alpha}\left(\frac{2s}{\alpha}\right)^{(0-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right), \\ &= \frac{\alpha^{j-1}}{2^j}\left(\frac{2s}{\alpha}\right)^{(2j-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right) \\ \frac{\partial^{j+k} \hat{\mathcal{K}}}{\partial \tau^j \partial \zeta^k}(u) &= (-1)^k \frac{\alpha^{j-1}}{2^j}\left(\frac{2s}{\alpha}\right)^{(2j+k-1)/2} \exp\left(-\zeta\sqrt{\frac{2s}{\alpha}}\right). \end{aligned} \tag{B.3}$$

Inverting (B.3), we obtain

$$\begin{aligned} \frac{\partial^{j+k} \mathcal{K}}{\partial \tau^j \partial \zeta^k} &= (-1)^k \frac{\alpha^{j-1}}{2^j}\left[\frac{\alpha}{(2\alpha\tau)^{j+k/2}} H_{2j+k}\left(\frac{u}{\sqrt{2}}\right)\right] \mathcal{K}(u), \\ &= \frac{(-1)^k \mathcal{K}(u)}{(4\tau)^j (2\alpha\tau)^{k/2}} H_{2j+k}\left(\frac{u}{\sqrt{2}}\right). \end{aligned} \tag{B.4}$$

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