

Medical Insurance after Retirement: Mathematical Models

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September 18, 2002

1 Introduction

At the first Hong Kong Industrial Applications Workshop held in the City University of Hong Kong, July 7-12, 2002, American International Assurance (AIA) presented a problem on medical insurance after retirement. Currently, Hong Kong residents enjoy one of the most generous public health care plans in the world. Most of the services are covered by public health insurance. Patients pay a small fraction of the costs for visits to doctors and hospital treatments. However, due to rising medical costs, declining government funding, and an aging population, the current public health plan will not be able to sustain and more medical services will be covered by the private sector in the future. It is likely that additional medical insurance will be needed to cover the rising costs, especially for those who will retire in the future. Based on this projection, AIA proposes a special medical insurance product, which covers the medical expenses of the clients only after retirement. The main question is, therefore, when an individual should start contribute toward this medical plan after retirement? In other words, if it is necessary to obtain such a plan, should an individual start the plan immediately, wait until he or she retires? If not, is there an optimal time to start contributing and how to determine the amount of the contribution?

In the following sections, we will propose several mathematical models which may be used to answer these questions. We will restrict ourselves to an idealized situation by assuming that the premium charged by the insurance will be used only to cover the actual medical expenses. We will neglect other direct and indirect costs as well as the profit.

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2 Deterministic Models

Due to the unique features of the insurance product being proposed by AIA, we first consider some simple deterministic models in this section. Our main objective is to investigate under what circumstances an optimal purchasing date exists.

Let's consider an individual of age a with annual income I who has to decide:

- a_* : the age to purchase a medical plan for coverage after retirement at age a_R ;
- c , the amount for consumptions (including living expenses, entertainment, etc.);
- s_R , total saving at retirement.

We assume that the amount of medical expenses after retirement (M) is given. We also assume that the premium of the insurance policy (m) depends on the health of an individual (pre-existing conditions) which is age-dependent. In particular, we assume that

$$m = p_c(a)\bar{m} \quad (1)$$

where \bar{m} is the premium without the pre-existing condition, and $p_c(a)$ is an increasing function of age a . We can view $p_c(a)$ as a penalty function which awards an early purchasing date and penalizes a late date.

2.1 Calculating premium \bar{m}

When the date a_* is given, we can calculate the premium $\bar{m}_* = \bar{m}(a_*)$ by balancing the accumulated premium and the total cost

$$\bar{m}_* + (1+r)\bar{m}_* + \dots + (1+r)^{a_R-a_*}\bar{m}_* = M \quad (2)$$

for a discretely compounded constant annual rate r and

$$\bar{m}_* + \bar{m}_* \int_{a_*}^{a_R} e^{r(a_R-t)} dt = M \quad (3)$$

for a continuously compounded constant annual rate r . The actual premium is then given by (1).

2.2 Zero interest rate cases

When we set $r = 0$, the total savings at the retirement for an individual of age a can be obtained as follows

$$s_R = (I - c)(a_R - a + 1) - [m(a_R) + m(a_R - 1) + \dots + m(a_*)] \quad (4)$$

where the second term on the right-hand-side is the accumulated sum of the premium and the discrete compounded rate is used. When $p_c(a) \equiv 1$, i.e., no penalty for pre-existing conditions, then the accumulated sum of the premium equals to M , thus (4) reduces to

$$s_R = (I - c)(a_R - a + 1) - M. \quad (5)$$

Note that a_* does not appear in the equation, the optimal purchasing age is arbitrary.

In general, we pay penalty for pre-existing conditions, i.e., $p_c(a) \geq 1$, we may want to maximize our savings s_R by choosing an appropriate a_* . From (4), we have

$$s_R = (I - c)(a_R - a + 1) - \frac{M}{a_R - a_* + 1} [p_c(a_R) + p_c(a_R - 1) + \dots + p_c(a_*)]. \quad (6)$$

It is not unreasonable to assume that $p_c(a)$ is a monotonically increasing function. Under this assumption, we have $a_* = a$, i.e., it is optimal (s_R reaches maximum) to start the plan as soon as possible.

2.3 Non-zero constant interest rate case

When the interest rate is a non-zero constant, we have

$$s_R = \frac{r(I-c)(a_R - a + 1)}{(1+r)^{a_R - a + 1} - 1} - \frac{rM}{(1+r)^{a_R - a_* + 1} - 1} [p_c(a_R) + (1+r)p_c(a_R - 1) + \dots + (1+r)^{a_R - a_*} p_c(a_*)]. \quad (7)$$

Again, without penalty for pre-existing conditions, the second term on the right-hand-side equals to M and a_* disappears from the equation. Thus, there is no reason for an earlier purchasing date. When penalty $p_c(a)$ exists and greater than unity, then an earlier purchasing date is preferred and $a_* = a$ will be optimal.

3 An optimal control formulation

We now consider a more realistic situation where an individual may die before retirement. Obviously it may not be an optimal choice to purchase the insurance immediately since one may not need the coverage. For the same reason, instead of s_R , we use a utility function as the objective function. We assume that

$$P(a, t) = e^{-\int_0^t \lambda(a+s) ds}$$

is the probability of an individual of age a survives until $a + t$ and $X(t)$ is the wealth at time t . Instead of maximize the saving at retirement, we will maximize the total utility over the life time of an individual. The rate of consumption c is no longer a pre-determined quantity. We will consider two cases distinguished by the investment strategy. In the first model, we consider only the risk-free investment while a combination of risk-free and risky investments is considered in the second model.

3.1 Risk-free investment

To simplify the situation, we assume that the wealth $X(t)$ increases as a result of fixed amount of annual income (e.g., salary) and the risk-free investment with constant interest rate r , subtracting the consumption and the premium for the medical insurance. The mathematical formula is

$$dX = f(a_*, c, X)dt \equiv rXdt - cdt + Idt - m(a_*)H(t - a_*)H(a_R - t)dt \geq 0 \quad (8)$$

with $X(a) = X_a$ as the initial wealth at age a . Here $H(t)$ is the Heaviside function. Our objective is to find c and a_* such that the total utility

$$V(a_*, c) \equiv \int_a^\infty F(t, a_*, c)dt$$

is maximized, i.e.,

$$\max_{a_*, c} V(a_*, c) \quad (9)$$

where

$$F(t, a_*, c) = e^{-r(t-a)} P(a, t) U(c)$$

and

$$U(c) = c^\gamma; \text{ or } \log(c).$$

3.2 General investment strategy

We now assume that the total investment can be divided into two portions:

- α , the amount for risky investments, e.g., stock;
- $1 - \alpha$, the amount for risk-less investments, e.g., government bonds.

The general framework is the same as before except that the increment for the wealth becomes

$$\begin{aligned} dX &= f(a_*, c, X)dt + g(a_*, c, X)dW \\ &\equiv (1 - \alpha)rXdt + \alpha\mu Xdt + \alpha\sigma XdW - cdt + Idt - m(a_*)H(t - a_*)H(a_R - t)dt \geq 0 \end{aligned} \tag{10}$$

with $X(a) = X_a$. Here dW is a Brownian motion with

$$E[dW] = 0; \text{ and } Var[dW] = 1.$$

4 Discussion

We have shown that by using a penalty function for the pre-existing conditions, an earlier purchasing date is preferred over a later one under the deterministic framework. By introducing mortality uncertainty, we derived two mathematical models based on which an optimal date may be calculated. However, due to the time constraint of the workshop, we are not able to further discuss these models in this report. Work is currently under way to investigate the existence of solutions to the two optimal control models, which will be presented elsewhere.

Acknowledgment. The author wishes to thank the organizers of the workshop at the City University of Hong Kong as well as American International Assurance for providing an excellent opportunity for working on an interesting problem.