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Visco-elastic Models of Asphalt Pavements

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This problem came from the Instituto Mexicano del Transporte, and was presented by Jose A. Romero-Navarrete.

The Institute has modeled the interaction between a truck moving at speed U and the asphalt pavement surface on which it travels. These simulations are intended to provide estimates of pavement wear as functions of parameters such as truck weight, speed, number of axles, tire size and pressure, time of day (temperature affects asphalt properties), etc. A current model has the truck represented by a multi-component system of masses, massless springs and dashpots, coupled by rigid connectors. The pavement is discretised into a number of sections ("tiles"), 30 cm. long, in the direction of travel.

The time over which a truck tire is in contact with a particular tile is $\Delta t = 30\text{cm.}/\text{U}$. During Δt the tire exerts a force on the tile surface which is transmitted through the asphalt layer, and which in turn provides an equal and opposite force on the tire footprint. In the current model the tile is also modeled as a spring and dashpot, and each tile moves independently of its neighbors. The tiles in contact with the truck tires are governed by ordinary differential equations, as is the truck system, and the motion may be found by solving the system over the time interval $0 \leq t \leq \Delta t$, given a set of initial conditions. In the interval $\Delta t < t < 2\Delta t$ the truck moves into contact with the next set of tiles, the system is solved again, and so on. The motion of the tiles under the moving load of the truck transmitted through the tire footprints then allows estimates of the energy absorbed by the pavement to be made, and this may be correlated with pavement wear.

Major simplifications present in the model described above are that a tile does not transmit information of its motion to neighboring tiles, and that the depth motion has a lumped parameter model description. The pavement is clearly more of a continuum than a set of independent discrete mechanical components, and satisfies a partial differential equation in the lateral and depth directions and in time. The problem posed to the workshop was to identify the continuum model appropiate to the motion of asphalt, to translate the model into a partial differential equation, to identify properties of its solution relevant to truck motion, and to indicate possible solution strategies based on this information.

In the following we examine visco-elastic models for the motion in the asphalt pavement. In order to keep the discussion at the simplest level we treat only one space dimension (depth). The lateral dimension(s) may be introduced later once the correct visco-elastic model has been identified to some satisfaction. In addition the Institute plans to conduct experiments to estimate parameters of the model chosen, and these experiments will consist of an impact force applied at the surface of a column of asphalt generating plane waves down the column.

One-dimensional Visco-elastic Model

The variables used to describe the motion are: $u(x,t)$, the location of a material element; $\varepsilon = \partial u/\partial x$ the strain experienced by that element; σ, the stress exerted by one element on another; ρ, the density of the material; I, the size of the impact pressure. The equation of motion is

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \tag{1}$$

$$\text{with } \sigma = -I\delta(t) \text{ at } x = 0, \tag{2}$$

$$\text{and } u = u_t = \sigma = \varepsilon = 0 \text{ at } t = 0 \text{ in } x > 0. \tag{3}$$

The impulse represented by (2) may be generalized subsequently to arbitrary stress inputs by convolution of the prescribed input with the solution to this system. In order to complete the system the material constitutive equation must be prescribed. In classical visco-elasticity theory the constitutive equation is a relation between σ and ε, and time derivatives of each. A general approach is (see [1]-[3]).

$$P\left(\frac{\partial}{\partial t}\right)\sigma = Q\left(\frac{\partial}{\partial t}\right)\varepsilon \tag{4}$$

where P, Q are polynomial functions. Particular examples are

(5) Elastic $P = 1, Q = E$

(6) Kelvin-Voigt $P = 1, Q = E + \eta\frac{\partial}{\partial t}$

(7) Maxwell $P = \frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}, Q = \frac{\partial}{\partial t}$

A four-element model has the Maxwell and Kelvin-Voigt elements in series. In the above, E is the elastic modulus and η the viscosity coefficient. Eliminating σ between (1) and (4) yields, for an elastic material

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad c^2 = E/\rho, \tag{8}$$

and for a Kelvin-Voigt material

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \frac{\eta}{\rho}\frac{\partial^3 u}{\partial x^2 \partial t}. \tag{9}$$

Maxwell and higher-order materials yield more complicated equations. Equation (8) is the standard wave equation; (9) is the wave equation plus a term representing the frictional effect.

Solutions to the system (1)-(4) are presented by Bland, [3], explicitly for the elastic and Maxwell materials and in accessible form for other materials. Materials separate into two classes depending on the form taken by the polynomials P, Q in (4):

Instantaneously Elastic Materials. These are the cases for which $\deg(Q) = \deg(P)$ as in the elastic and Maxwell materials. Here the system has hyperbolic properties, with the solution for the stress impact represented by (2) exhibiting a wave-front propagating at speed c. The solution ahead of the wave-front, that is in $x > ct$, is zero, and there is a disturbance left behind the wave-front (described by a Bessel function for the Maxwell case) which decays in time if there is viscosity present. Bland presents results for general materials valid near the wave front in terms of parameters obtained from the asymptotic expansion of $P(i\omega)/Q(i\omega)$ for $\omega \gg 1$, where ω is the frequency parameter related to the Fourier transform in t.

Materials Not Instantaneously Elastic. Here $\deg(Q) = \deg(P) + 1$ as is the Kelvin-Voigt material. The system now has parabolic properties and the disturbance is non-zero in $x > 0$ for $t > 0$ (it may be exponentially small as $x \to \infty$). Bland presents the solution for a general material for t small, again in terms of $P(i\omega/Q(i\omega)$ for $\omega \gg 1$: it has the characteristic Gaussian profile arising in diffusion problems. The material is acting like a viscous fluid. In the case of the Kelvin-Voigt material, for example, the viscous term in (6) will be dominant under the impulse (2), giving stress proportional to rate-of-strain, the constitutive equation for a viscous fluid.

The discussion above delineating solution properties in terms of the type of material model chosen indicates the importance of correct specification of the model. Solutions to (8) and (9) are quite distinct even for $\eta \ll 1$, and the differences grow in time. Sharp fronts evident for (8) are not present for (9): a paradigm is the difference between $\mathrm{sgn}(x)$ and $\tanh\,(x/\eta)$. Experiments to identify the visco-elastic model to be adopted need to be designed carefully so that critical measurement variables are chosen, and subtle differences in model are not lost in the noise of measurement error.

Comments on Solution Strategy

Once the material model has been chosen an appropriate boundary-value problem must be determined. Two simplifications are suggested here, each based on reducing the dimensionality of the problem suggested by the occurrence of small parameters.

(a) The asphalt layer, typically 1m. thick, sits on a layer of crushed stone above graded earth. Motions created at the surface of the half-space will undergo viscous damping as they travel and will probably be weakly reflected at the asphalt-stone interface. This implies that for calculations involving the mutual interaction of two tires situated some meters apart, the waves from one tire experienced by the other will have travelled along the surface of a thin strip, a ribbon of visco-elastic material sitting on a low reflectance base. Only the lateral space variables will occur in the partial differential equation, (the three dimensional equivalent of (1)) and the effect of depth will be seen only in terms of energy leakage - similar to the telegrapher's equation.

(b) The replacement for the boundary-value problem (1)-(4) to include lateral space dimensions will have a travelling stress input $\delta(t-z/U)$ in (2). If the lateral space dimension z is discretized (tiled) in increments Δz, the contact time for a tire on a sigle tile is

$$\Delta t = \Delta z/U. \tag{10}$$

Consider also the time T taken for a wave to travel between two tires a distant L apart:

$$T = L/c. \tag{11}$$

The ratio of these times

$$\epsilon = \frac{T}{\Delta t} = \frac{L}{\Delta z}\frac{U}{c}, \tag{12}$$

calculated for typical values ($\Delta z = 30$cm., $L = 10$m., $U = 30$m./sec, $E = 3.10^9$Pa., $\rho = 2.3.10^3$kg./m.^{3}, $c = 1.14.10^3$m./sec) has the value $\varepsilon = 3/4$. This means that in the time taken for the wave created by one tire to reach a second tire 10 m. away, the latter has moved only 3/4 across its tile. To a large extent, therefore, the tires could be frozen in place on their respective tiles during the calculation of mutual interaction of tire motions via the waves they are creating. This may reduce the size of the computation necessary. Of course for this to hold ϵ must be sufficiently small, and this determines bonds on the parameters appearing in (12) if different from values assumed here.

Consideration of the **damping** effect on a wave traveling between two tires a distance L apart also implies small interaction between the two tire motions if $L \sim 10$m. For a Maxwell material the wave undergoes damping proportional to $\exp(-Et/2\eta)$. Typical values for $L = 10$m., $c = 10^3$m./sec give $t = 10^{-2}$sec. Also $E = 10^9$ Pa. and η is in the range 10^6 to 10^7 Pa. sec. The damping ranges from $\exp(-15)$ to $\exp(-3/2)$ for these values. In addition there will be spatial decay. The conclusion is that wave calculations will be necessary only for tires in close proximity.

Summary

Analysis of the solutions for waves created by an impulse shows that the nature of these waves depends critically on the visco-elastic model adopted. Preliminary estimates of wave damping indicate that waves from one tire will be negligible at another tire 10 m. away. The time scales indicate also that the waves propagate at speeds sufficiently large that the tires appear stationary in the calculations for the effect of the waves from one tire on the motion of another. These considerations will substantially reduce the amount of calculation needed to extend the simulations to the case where the pavement is a continuum.

References

[1] Flügge, Wilhelm, *Viscoelasticity*, (2nd ed), Springer-Verlag, New York, 1975.

[2] Mase, G. E., *Theory and Problems of Continuum Mechanics, Schaum's Outline Series*, McGraw-Hill, New York, 1970.

[3] Bland, D. R., *The Theory of Linear Viscoelasticy*, Pergamon Press, Oxford, 1960.