Study Group Report
The ICI Airbag Problem

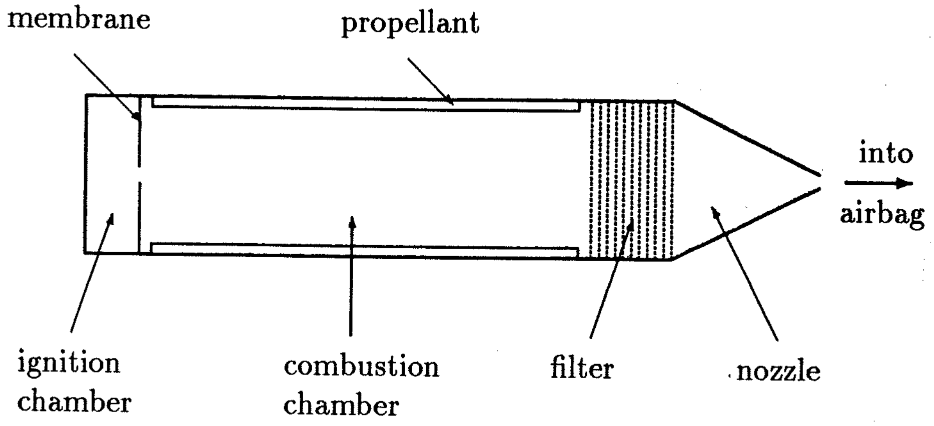


Figure 1: Schematic diagram of apparatus.

1 Introduction

We were asked to investigate the system of airbag inflation shown in Figure 1. From left to right, the apparatus can be divided into four regions:

Region 1

An igniter releases hot gas which ruptures a membrane at a high temperature and pressure.

Region 2

This gas flows into a combustion chamber, igniting a propellant (sodium azide) which partially coats the upper and lower surfaces of the chamber. The propellant then continues to burn for a while, releasing gas and some solid particles of combustion products into the chamber.

Region 3

The hot gas and particles enter a filter where, it is hoped, all the solid particles are captured. The filter may also absorb some heat from the gas flow. The filter is composed of many layers of coarse and fine wire, steel wool and inorganic fibre.

Region 4

The gas and any remaining particles then travel through a converging nozzle and into an airbag which inflates.

System dimensions and typical values for the process are given in the following table.

Combustion chamber length	115mm
Combustion chamber pressure	10^7 Pa
Combustion chamber temperature	10^3 K
Filter length	5mm
Gas velocity at filter	5ms $^{-1}$
Time to airbag inflation	30ms

The questions posed by ICI were

1. What is the gas flow through the filter and nozzle?
2. How are particles trapped in the filter?
3. What heat transfer occurs in the filter?
4. Given the pressure outside the nozzle, can boundary conditions for the downstream end of Region 2 be given, and can the volume flux of gas through the downstream end of the nozzle be predicted? Note that ICI have a model for the flow in Region 2.

Attempts were made to address all of these questions but limited time meant that it was not possible to consider properly all aspects of the problem. Little time was spent on the question of heat transfer within the filter.

The strategy we adopted to answer the fourth question was to formulate separate models for the combustion chamber, the filter and the nozzle. An estimate of the initial conditions can be made and each model solved in turn to provide boundary conditions at the upstream end of the next region.

In essence, when brief transients arising from the ignition (and extinction) of the combustion are ignored, we believe that the pressure at the point where gases and solid particles enter the filter determines, and is determined by, conditions at the outlet. These conditions depend on whether or not the nozzle flow is choked—that is whether sonic flow occurs at the point of maximum constriction. If it is choked (a condition that can always be achieved by making the nozzle cross-sectional area small enough) then the operation of the apparatus depends only on the design of the apparatus itself and is uninfluenced by the outside pressure. If it is not choked then an external condition (such as the stagnation pressure) determines the operation of the apparatus. It seems that choking is desirable to ensure a high pressure in the combustion chamber, and hence a high gas production rate.

The operation may change with time as the filter becomes progressively 'clogged up' with trapped particles or as the filter heats up and so removes progressively less heat from the flow. A catastrophic change in behaviour could arise if the filter were to become completely clogged up, stopping all gas flow through it. These changes can be illustrated using a much simplified filter model.

2 The Combustion Chamber

Although ICI informed us that they have a numerical model for the combustion chamber, it is important to have at least a broad idea of the operation of this region in order to place the filter and nozzle analyses in context. We begin with a semi-empirical discussion.

Processes in the combustion chamber

Ignition:

The membrane of the ignition chamber is designed to burst at a pressure of about 10^3 atmospheres. This enormous pressure results in the propagation of a very strong shock wave, which, according to the strong-shock limit, has pressure and temperature ratios across it related by

$$\frac{p}{p_i} = \frac{\gamma+1}{\gamma-1}\frac{T}{T_i}. \tag{1}$$

Where subscript i refers to initial (or undisturbed) conditions. Although adiabatic expansion of the gases behind the shock rapidly reduces the post-shock pressure in proportion to $(V/V_i)^{-\gamma}$, V being the specific volume, a polytropic exponent of $\gamma = 1.4$ (for air) predicts an initial post-shock temperature of about $50\,000\,°K$! Complicated shock reflections at the filter and nozzle may tend further to enhance the transient temperature field.

This very hot (shocked) air is rapidly convected out of the apparatus. However, together with the following hot gases (initially produced in the igniter at about $2\,000\,°K$) it is likely to aid the ignition of the sodium azide propellant on the walls of the combustion chamber. Lack of information about the process meant that this ignition transient could not be studied, although one may safely assume that it is likely to be of short duration.

Combustion:

ICI offered us a 'steady' free-mass (i.e. gas and solid particles) production rate of the form

$$\dot{m} = ap^b \tag{2}$$

where $b \approx 0.3$ and a depends on the total surface area of burning propellant. The problem cannot be closed however without also knowing the temperature of the combustion products. In determining this, we consider that a thin slowly-propagating combustion wave in the sodium azide converts the propellant from solid at room temperature into solid and gas at a higher temperature, with a fixed change in energy resulting from the chemical change. Because the gases emanating from the wave travel at a low Mach number, the kinetic energy of the reaction products can be neglected and the conversion can be considered to take place at constant pressure (see below). Thus, with fixed specific heats at constant volume of each component, and subscripts s and g referring to solid and gas phases respectively

$$C_{vsi} T_i + Q = (C_{vs} + C_{vg}) T_B + p \Delta V \tag{3}$$

for a heat of reaction Q. Neglecting the volume of the solid-phase, the change in specific volume is obtainable from the ideal gas law: $\Delta V \approx V_g = RT_B/p = (C_{pg} - C_{vg}) T_B/p$. Hence the 'burnt' temperature becomes

$$T_B = \frac{C_{vsi} T_i + Q}{C_{vs} + C_{pg}} \tag{4}$$

which is fixed, regardless of the pressure. Allowing specific heats to change, one may postulate a more general burnt-temperature law

$$T_B = T_B(p) \tag{5}$$

which may be a weak function of pressure, as is the steady burning law (2). However, we expect T_B to remain more-or-less a fixed constant.

Quasi-steady burn:

We note that the velocity during steady burning in the combustion chamber is of the order of 5 metres per second, giving a Mach number M of less than about 10^{-2}. It follows that pressure changes across the combustion chamber, which are such that

$$\frac{\Delta p}{p} \approx \tfrac{1}{2}\gamma M^2 \approx 10^{-4}, \tag{6}$$

are very small indeed. Taking ρ to represent the overall mixture density of gas and solid particles together, and again neglecting the volume of solids, the ideal gas law gives

$$p = R(1 - \mu)\rho T \tag{7}$$

where, for all practical purposes, $T = T_B$, and where μ represents the mass fraction of the solid ($\mu \approx 0.3$). R is the gas constant for the gaseous products of the combustion. If $\dot{m}(p)$ is the total mass production rate of gas and particles, and if these are flowing at the same uniform velocity u across a cross-sectional area A at a position x along the length of the combustion chamber, the steady mass flux becomes

$$\frac{x}{L}\dot{m}(p) = A\rho u \qquad \text{so that} \qquad u = \frac{x\dot{m}}{AL}\frac{R(1-\mu)T}{p} \tag{8}$$

where L is the length of the combustion chamber. At the end of the chamber, $x = L$ (i.e., at the beginning of the filter), the averaged velocity over the filter area is

$$u = \frac{\dot{m}}{A}\frac{R(1-\mu)T}{p} \tag{9}$$

where in more complex geometries A may simply be taken to denote the area of the filter.

Simple model:

In an averaged, one-dimensional system, in which the relaxation time over which the velocity of a particle adjusts towards that of the gas surrounding it, is negligibly small, the continuity and momentum equations are

$$\rho_t + (\rho u)_x = \dot{m}/(AL) \quad \text{and} \quad \rho(u_t + uu_x) + p_x = 0. \tag{10}$$

In order to complete this set, one needs only to invoke a law such as the adiabatic law

$$p\rho^{-\Gamma} = \text{const.} \qquad \text{where} \quad \Gamma = \frac{(1-\mu)C_{pg} + \mu C_{vs}}{(1-\mu)C_{vg} + \mu C_{vs}} \in (1, \gamma], \tag{11}$$

which applies if compressive heating penetrates equally rapidly into both gas and particle phases. Solutions of these equations may be separated into the components

$$u = \frac{x}{L}\frac{\dot{m}}{A\rho} + \tilde{u}(x,t) \quad \text{and} \quad p = \mathcal{P}(t) - \tfrac{1}{2}\rho u^2 + \tilde{p}(x,t), \tag{12}$$

where $\tilde{u}(x,t)$ and $\tilde{p}(x,t)$ represent acoustic or gasdynamic waves which, when set to zero, duplicate the simple quasi-steady description (8). After initial transient processes, the dominant pressure component is the spatially uniform term $\mathcal{P}(t)$ which may change significantly on a time-scale over which the effect of the filter or the external pressure (if the flow is not choked) may change significantly.

Transients:

In order to appreciate the role of the quasi-steady solutions, it remains to identify the appropriate time-scales associated with acoustic waves, quasi-steady flow and changes in the filter or external environment.

Based on the description given for the problem, we are able to suppose that acoustic waves traverse the apparatus in less than $\frac{2}{10}$ ms, the speed of sound in the combustion chamber being about 700 metres per second. The approximate time taken for a molecule of combustion products to be swept out of the apparatus by the flow is of the order of 10 to 20 ms. For practical reasons, it should be necessary to design the apparatus such that the time-scale over which significant changes in the operation of the filter (especially clogging) would be longer than the overall burning time of the propellant (say 50 to 100 ms). If this were not so, a potentially catastrophic blockage in the flow out of the apparatus might occur.

The relatively very short acoustic time means that acoustic waves are able to travel and to reflect many times across the combustion chamber. At least at the filter and nozzle, where some acoustic energy can be transmitted through to the airbag, reflections involve a loss of amplitude so that an acoustic (or gasdynamic) wave is greatly diminished after several reflections. On the other hand, slow changes due to the filter or external conditions manifest themselves as acoustic waves in the chamber of very small amplitude.

In essence, it can be argued that the quasi-steady almost uniform-pressure solution $p \approx \mathcal{P}(t)$ dominates the behaviour in the combustion chamber, and is fully determined by the pressure at the inlet to the filter. In turn, this solution determines the mass-production rate $\dot{m} \approx \dot{m}(\mathcal{P})$, the velocity at the filter inlet (via equation (9)) and the temperature at the filter inlet $T \approx T_B$.

One dimensional two-phase model

Although ICI already have a model for this part of the process, we thought it worthwhile to construct a simple model of our own which could be solved numerically.

The gas and solid particles are assumed to behave as separate phases and the flow is taken to be one-dimensional. The gas is assumed to be perfect and the gas flow to be adiabatic. The burning of the propellant produces solid and gas and so that source terms are included in the gas and solid mass and momentum equations (in the latter, these terms arise only through the use of a conservative form for the equations; there is, of course, no chemical production of momentum). We also include interfacial forces between the gas and the solid. The model equations are[1]

$$\begin{aligned}
(\alpha_g \rho_g)_t + (\alpha_g \rho_g u_g)_x &= \dot{m}_g(p), & (13)\\
(\alpha_s \rho_s)_t + (\alpha_s \rho_s u_s)_x &= \dot{m}_s(p), & (14)\\
(\alpha_g \rho_g u_g)_t + (\alpha_g \rho_g u_g^2)_x &= -\alpha_g p_x + u_g \dot{m}_g + F_{gi}, & (15)\\
(\alpha_s \rho_s u_s)_t + (\alpha_s \rho_s u_s^2)_x &= u_s \dot{m}_s + F_{si}, & (16)\\
\alpha_s + \alpha_g &= 1, & (17)\\
p &= \kappa \rho_g^\gamma, & (18)\\
p &= \rho_g RT. & (19)
\end{aligned}$$

Here, x is distance measured from the membrane, t is time, p is pressure and T temperature. The subscripts g and s denote the gas and solid phases respectively; the α_i's are the volume fractions of the phases, the u_i's their velocities and the ρ_i's their densities.

In equations (13) and (14), $\dot{m}_g$ and $\dot{m}_s$ are the rate of mass production (per unit volume per unit time) of gas and solid respectively. ICI informed us that $\dot{m}_g$ and $\dot{m}_s$ depend only on p and that the propellant yields 18% free solid, 42% gas and 40% residue (by mass) so that

$$\dot{m}_g = \frac{7}{3}\dot{m}_s = 0.7 \times \dot{m}(p)/(AL).$$

[1] An alternative set of one dimensional model equations follows. In this, we use the volume fraction of solid $\alpha = \alpha_s$ to identify overall densities of solid and gas, measured in terms of mass per unit volume of the two-phase mixture, as $\varrho_s = \alpha \rho_s$ and $\varrho_g = (1-\alpha)\rho_g$. Note that $\rho = \varrho_s + \varrho_g$ and $\mu = \varrho_s/\rho$.

$$\varrho_{st} + (\varrho_s u_s)_x = \mu_0 \dot{m}(p)/(AL)$$

$$\varrho_{gt} + (\varrho_g u_g)_x = (1-\mu_0)\dot{m}(p)/(AL)$$

$$\varrho_s(u_{st} + u_s u_{sx}) + \alpha p_x = (\varrho_s + \varrho_g)(u_g - u_s)/\tau_u$$

$$\varrho_g(u_{gt} + u_g u_{gx}) + (1-\alpha)p_x = (\varrho_s + \varrho_g)(u_s - u_g)/\tau_u$$

$$\varrho_s C_{vs}(T_{st} + u_s T_{sx}) = (\varrho_s C_{vs} + \varrho_g C_{pg})(T_g - T_s)/\tau_T$$

$$\varrho_g C_{pg}(T_{gt} + u_g T_{gx}) = p_t + u_g p_x + (\varrho_s C_{vs} + \varrho_g C_{pg})(T_s - T_g)/\tau_T$$

$$p = RT_g \varrho_g/(1-\alpha) \qquad \text{and} \qquad \alpha = \varrho_s/\rho_s.$$

The simple model (10) and (11) can be derived from these equations in the limits as the momentum and thermal relaxation times τ_u and τ_T, and the volume fraction α, tend to zero. This model explicitly considers thermal transfer and pressure-enthalpy relations instead of more simplistic adiabaticity of the gas phase alone. The terms μ_0, AL, τ_u, τ_T, C_{vs}, C_{pg}, ρ_s and R may be treated as constants.

In equations (15) and (16), F_{gi} and F_{si} are the interfacial 'forces' (per unit volume) acting on the gas and solid respectively. For momentum to be conserved, $F_{gi} = -F_{si}$. In (16), we have assumed that the gas is 'dusty', that is that the particles are small and far apart. Thus, their interaction with each other is negligible and we do not have to include a solid pressure term.

In equation (18), κ is the entropy of the gas and the ratio of the specific heats, γ is about 1.3. In equation (19), R is the universal gas constant divided by the molecular weight of the gas; $R \approx 277$. The solid density, ρ_s, is constant and, since $\alpha_s \ll \alpha_g$, we take $\alpha_g = 1$.

The set of model equations given in the footnote on Page 6 are equivalent to equations (15)–(19) except that the latter assume a Stokes drag between the particles and gas, and include a more realistic model for heat transfer than the assumption in equation (18) of adiabatic flow.

We define dimensionless variables $\hat{x}$, $\hat{t}$, $\hat{\rho}_g$, $\hat{u}_g$, $\hat{u}_s$, $\hat{\alpha}_s$, $\hat{T}$ and $\hat{p}$ as follows;

$$
\begin{array}{lll}
x = L\hat{x}, & t = \frac{L}{U_g}\hat{t}, & \rho_g = \rho_0\hat{\rho}_g, \\
u_g = U_g\hat{u}_g, & u_s = U_s\hat{u}_s, & \alpha_s = \alpha_0\hat{\alpha}_s, \\
T = T_0\hat{T}, & p = \rho_0 R T_0 \hat{p}. &
\end{array}
$$

Here, L is the combustion chamber length, U_g and U_s are typical gas and solid velocities, ρ_0 is a typical gas density, α_0 a typical particle volume fraction and T_0 a typical temperature. The choice of $\hat{t}$ as measuring flow times anticipates convective transients to be present. As noted earlier, boundary conditions at the filter are expected to change more slowly than this.

Substituting into equations (1) to (7) and dropping the hats for convenience, gives the following set of dimensionless equations:

$$(\rho_g)_t + (\rho_g u_g)_x = \dot{m}_g^*(p), \tag{20}$$
$$(\alpha_s)_t + (\alpha_s u_s)_x = \dot{m}_s^*(p), \tag{21}$$
$$(\rho_g u_g)_t + (\rho_g u_g^2)_x = -\lambda p_x + u_g \dot{m}_g^* + F_{gi}^*, \tag{22}$$
$$(\alpha_s u_s)_t + (\alpha_s u_s^2)_x = u_s \dot{m}_s^* + F_{si}^*, \tag{23}$$
$$p = \kappa^* \rho_g^\gamma, \tag{24}$$
$$p = \rho_g T, \tag{25}$$

where the dimensionless mass production rates and dimensionless forces are defined by

$$
\dot{m}_g = \frac{\rho_0 U_g}{L}\dot{m}_g^*, \qquad \dot{m}_s = \frac{\alpha_0 \rho_s U_s}{L}\dot{m}_s^*,
$$
$$
F_{gi} = \frac{\rho_0 U_g^2}{L}F_{gi}^*, \qquad F_{si} = \frac{\alpha_0 \rho_s U_s^2}{L}F_{si}^*,
$$

and

$$
\kappa^* = \frac{\kappa}{RT_0}, \qquad \lambda = \frac{R_0 T_0}{U^2}.
$$

Note that $\lambda \gg 1$ and we also expect λ to be much greater than a characteristic value of F_{gi}^*, u_g and u_s being almost equal. Thus, spatial pressure variations will be small compared to the large ambient pressure in the chamber, as discussed earlier.

This system of hyperbolic equations with suitably estimated initial and boundary conditions could be solved numerically. However, it may be noted that the numerical problem is likely to be stiff due to the presence of relatively high frequency acoustic waves.

3 The Filter

As in the combustion chamber, we use the equations of one-dimensional two-phase flow. Equivalently, we may think of the mathematical formulation as describing a flow with three phases: (*i*) the free gas, (*ii*) the free solids and (*iii*) the immobile solid matrix. [2] The porosity, φ, of the filter may change significantly during the time that it takes the airbag to inflate. Our calculations indicate that the average porosity may decrease as much as 50% below its initial value of about $\frac{1}{2}$. Of course, it is the permeability of the filter that is crucial in the present context. Even though the value of the avarage porosity may be $\frac{1}{4}$ at the end of the process, the particle capture could (for example) be concentrated at the front face of the filter, thus creating an impenetrable blockage. The filter design must take account of this and aim to minimise the overall resistance to the flow. A realistic mathematical model should therefore include equations for particle capture and consequential changes in permeability.

The Reynolds number of the gas flow through the filter, based on the largest 'grain' size, is about 10, and so it seems reasonable to extrapolate the well-known result for particulate porous media to 'fibrous' media and use a Darcy drag law, $D = ku_g/\mu$ in dimensional terms. Here μ is the coefficient of viscosity of the gas and k is the permeability of the porous medium. One possible permeability–porosity relationship is given by the Kozeny-Carman equation—though this is not wholly satisfactory, since the result is based on a capillary-tube model for the porous medium.

In what follows, we take the permeability to be constant in order to simplify analytical progress. The variables are non-dimensionalized with reference to the same values as in the combustion chamber except that the spatial co-ordinate x is non-dimensionalized with the filter length, d, rather than the combustion chamber length, L. As in the combustion chamber, we take $\alpha_g = 1$. The dimensionless model equations are

$$\frac{d}{L}(\varphi\rho_g)_t + (\varphi\alpha_g\rho_g u_g)_x = 0, \tag{26}$$

$$\frac{d}{L}(\varphi\alpha_s)_t + (\varphi\alpha_s u_s)_x = -K\alpha_s u_s \varphi l(\varphi), \tag{27}$$

$$\frac{d}{L}(\varphi\rho_g u_g)_t + (\varphi\rho_g u_g^2)_x = -\varphi\lambda p_x + \frac{d}{L}\varphi F_{gi}^* - \varphi A\rho_g u_g, \tag{28}$$

$$\frac{d}{L}(\varphi\alpha_s u_s)_t + (\varphi\alpha_s u_s^2)_x = -K\alpha_s u_s^2\varphi l(\varphi) + \frac{d}{L}\varphi F_{si}^*, \tag{29}$$

$$\varphi_t = -\beta\alpha_s u_s \varphi l(\varphi), \tag{30}$$

$$p = \rho_g T, \tag{31}$$

[2] A more sophisticated approach would be to use continuum theory of mixtures or local volume averaging. See, for example, M.S. Willis (1983) "A multiphase theory of filtration" in Progress in Filtration and Separation, Vol. 3, edited by R.J. Wakeman, Elsevier.

In equations (27) and (29), $(K/d)l(\varphi)$ is the fractional volume of solid particles which are deposited on the filter, per unit length of filter. We expect that $l'(\varphi) < 0$ and, for simplicity, take $l(\varphi) = 1 - \varphi$. In equation (30), $\beta/\alpha_0 L$ is the fraction of filter volume clogged up per unit volume of particles deposited. In equation (28), the dimensionless drag coefficient $A = d\mu/(\rho_g u_g K)$.

In general, we also need a heat conservation equation. At the Study Group meeting, however, there was not enough time to consider the heat transfer problem for the filter. For simplicity, it was assumed that the gas flow is adiabatic, so that T is solely a function of time determined by the conditions at the exit from the combustion chamber.

We make the following simplifications to equations (26) to (31):

1. Since $d/L \ll 1$, we may neglect time derivatives in comparison to space derivatives in equations (26) to (29). Thus transients are neglected, not only on the longer 'clogging' time-scale but also on the time-scale of convective flow in the combustion chamber.

2. We expect that $|F_{gi}| \ll L/d$ and A is order 1 or larger, so that the interaction between the gas and the particles in equation (28) can be neglected in comparison to the interaction between the gas and the filter.

3. Likewise, we expect that $|F_{gs}| \ll L/d$ and that $K = O(1)$ so that the interaction between the particles and the gas in equation (29) can be neglected in comparison to the interaction between the particles and the filter.

With these simplifications, equations (26) - (31) become,

$$(\varphi\rho_g u_g)_x = 0, \quad (32)$$

$$(\varphi\alpha_s u_s)_x = -K\alpha_s u_s \varphi(1-\varphi), \quad (33)$$

$$(\varphi\rho_g u_g^2)_x = -\varphi\lambda p_x - \varphi A\rho_g u_g, \quad (34)$$

$$(\varphi\alpha_s u_s^2)_x = -K\alpha_s u_s^2 \varphi(1-\varphi), \quad (35)$$

$$\varphi_t = -\beta\alpha_s u_s \varphi(1-\varphi), \quad (36)$$

$$p = \rho_g T. \quad (37)$$

Note that the solid and gas phases are coupled through their interaction with the filter.

Assuming that the problem in the combustion chamber has been solved, the gas density, solid volume fraction and volume flux of particles and gas are all prescribed at the inlet to the filter. Thus,

$$\begin{aligned} \alpha_s &= a(t) && \text{at} \quad x = 0, \\ \varphi u_s &= b(t) && \text{at} \quad x = 0, \\ \rho_g &= c(t) && \text{at} \quad x = 0, \\ \varphi u_g &= g(t) && \text{at} \quad x = 0. \end{aligned}$$

Solution of the Filter Equations

To enable analytical progress to be made, we assume that the filter is uniform so that

$$\varphi = \varphi_0 \qquad \text{at} \quad t = 0.$$

where φ_0 is the initial porosity of the filter, and that A is a constant. We discuss later how allowing for non-uniformities in the filter porosity affects the analysis.

Eliminating $\alpha_s u_s$ between equations (33) and (36), gives the following partial differential equation for φ

$$\left[\frac{\varphi_t}{1-\varphi}\right]_x = -K\varphi_t \tag{38}$$

with $\varphi = \varphi_0$ at $t = 0$. This can be integrated to give

$$\varphi = \frac{\varphi_0 - B(t)\exp[-K(1-\varphi_0)x]}{1 - B(t)\exp[-K(1-\varphi_0)x]}, \tag{39}$$

where $B(0) = 0$.

To determine $B(t)$, we substitute for φ into equation (36) and use the boundary conditions on α_s and φu_s at $x = 0$. This gives

$$\frac{B'(t)}{1-B(t)} = \beta a(t)b(t),$$

so that

$$B(t) = 1 - \exp\left\{-\beta\int_0^t a(\hat{t})b(\hat{t})\,d\hat{t}\right\}. \tag{40}$$

From equations (33) and (35), u_s is independent of x and thus, using the boundary conditions at $x = 0$,

$$u_s = \frac{b(t)(1-B(t))}{\varphi_0 - B(t)}. \tag{41}$$

We may now determine α_s from equation (36);

$$\alpha_s = \frac{B'(t)\exp[-K(1-\varphi_0)x](\varphi_0 - B(t))}{\beta b(1-B(t))\{\varphi_0 - B(t)\exp[-K(1-\varphi_0)x]\}}. \tag{42}$$

Note that the 'clogging time', t_c, at which $\varphi \to 0$ at $x = 0$, $u_s \to \infty$ and $\alpha_s \to 0$ is given by

$$\int_0^{t_c} a(\hat{t})b(\hat{t})\,d\hat{t} = \frac{1}{\beta}\log\left(\frac{1}{1-\varphi_0}\right). \tag{43}$$

The solutions for φ and α_s are plotted for various times in Figures 2 and 3 taking $\varphi_0 = 0.5$. For simplicity, we have taken the flow at the inlet to the filter to be uniform.

Turning to the gas momentum equation, equation (34), we note that $\lambda \gg 1$ and, we expect, $\lambda \gg A$. Thus, we expect the pressure drop across the filter to be small compared to the ambient pressure.[3] Thence, from equations (37) and (32), ρ_g and φu_g will be approximately constant. To find the pressure drop across the filter, we write

$$\begin{aligned} p &= c(t)T(t) + \epsilon p_1 + O(\epsilon^2), \\ \rho_g &= c(t) + \epsilon \rho_{g1} + O(\epsilon^2), \\ u_g &= \frac{g(t)}{\varphi} + \epsilon u_{g1} + O(\epsilon^2), \end{aligned}$$

[3] Note that this assumption will break down as the clogging time is approached. In equation (34), the inertia terms will become comparable with the pressure gradient when $\varphi \sim 1/\lambda$. In this case, the pressure drop could be found by eliminating between equations (32), (37) and (34) to give a nonlinear ordinary differential equation for p which could be solved numerically.

where $\epsilon = A/\lambda \ll 1$. Then, from equation (34),

$$p_{1x} = -\frac{g(t)c(t)}{\varphi} - \frac{1}{A}g^2(t)c(t)\frac{\phi_x}{\phi^3}$$

so that

$$\begin{aligned} p_1 &= -g(t)c(t)\int_0^x \frac{d\hat{x}}{\varphi(\hat{x},t)} - \frac{1}{2A}g^2(t)c(t)\left\{\left(\frac{1-B}{\phi_0 - B}\right)^2 - \frac{1}{\phi^2}\right\} \\ &= -g(t)\left\{x + \frac{1-\varphi_0}{K}\log\left[\frac{\varphi_0 \exp[K(1-\varphi_0)]x - B}{\varphi_0 - B}\right]\right\} \\ &\quad -\frac{1}{2A}g^2(t)c(t)\left\{\left(\frac{1-B}{\phi_0 - B}\right)^2 - \frac{1}{\phi^2}\right\}. \end{aligned} \tag{44}$$

Thus the dimensional pressure drop across the filter, p, is given by

$$p = \rho_0 d\hat{A}U_g g(t)c(t)\left\{\int_0^d \frac{dx}{\varphi(x,t)} + \frac{g(t)d}{2A}\cdot\left[\frac{1}{\phi(x,t)^2}\right]_{x=0}^{x=d}\right\}. \tag{45}$$

The pressure is plotted for various times in Figure 4, again taking $\varphi_0 = 0.5$.

4 The Nozzle

We assume that the mass fraction of particles leaving the filter is negligible. Note that our filter model does not allow us to prescribe that all the particles are removed by the filter, but if $K(1-\varphi_0) \gg 1$ then the number of particles entering the nozzle will be negligible and we may assume one-dimensional inviscid compressible adiabatic gas flow there. Mass conservation, Bernoulli's equation and the adiabatic condition imply that the gas velocity, u_4, and density, ρ_4 at the nozzle outlet satisfy

$$A_4\rho_4 u_4 = A_3\rho_3 u_3, \tag{46}$$

$$\frac{\gamma-1}{2}u_4^2 + a_4^2 = \frac{\gamma-1}{2}u_3^2 + a_3^2, \tag{47}$$

$$p_4/\rho_4{}^\gamma = p_3/\rho_3{}^\gamma \tag{48}$$

where u_3 and ρ_3 are the gas density and velocity at the nozzle inlet or filter outlet, A_3 and A_4 are the inlet and outlet cross-sectional areas and the sound speeds at these positions are

$$a_j^2 = \gamma p_j/\rho_j, \qquad j = 3, 4.$$

Since u_3, p_3 and ρ_3 are known from the solution of the filter model, u_4, p_4 and ρ_4 can be found. There is certainly a possibility that the Mach number may reach unity, so that the flow would be choked at the nozzle outlet. Whether or not this happens is essentially determined by the constriction ratio A_4/A_3. There are therefore two distinct forms of final boundary condition that determine the overall behaviour of the apparatus:

$$\text{either} \qquad u_4 = a_4 \tag{49}$$

$$\text{or} \qquad p_3 + \frac{\gamma-1}{2\gamma}\rho_3 u_3^2 = p_3^{1/\gamma}p_\infty^{(\gamma-1)/\gamma} \tag{50}$$

for a specified 'external' stagnation pressure $p_\infty(t)$ which would normally be of the order of magnitude of the atmospheric pressure.

Because the flow Mach number is very low at the exit from the filter, the latter (unchoked) condition is very closely approximated (to an accuracy of about 0.01%) by

$$p_3 \approx p_\infty. \tag{51}$$

Since the pressure drop across the filter is unlikely to be large (unless it is clogged), this condition therefore shows that, under unchoked conditions, the combustion chamber pressure would not normally differ greatly from atmospheric pressure. This situation certainly does not normally hold, pressures being of the order of 100 atmospheres in the combustion chamber. Indeed it is probably highly undesirable. One must therefore conclude that choking is probably the most desirable operating condition and the constriction ratio A_4/A_3 would need to be designed accordingly. For this purpose, one may note from equations (46)–(49) that

$$p_3 + \frac{\gamma - 1}{2\gamma}\rho_3 u_3^2 = p_3^{2/(\gamma+1)} \times \frac{\gamma+1}{2}\left[\left(\frac{A_3}{A_4}\right)^2 \frac{\rho_3 {u_3}^2}{\gamma}\right]^{\frac{\gamma-1}{\gamma+1}} \tag{52}$$

in which the term containing u_3 on the left hand side may reasonably be neglected, as in (50), to yield simply

$$p_3 \approx \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} \left(\frac{A_3}{A_4}\right)^2 \frac{\rho_3 {u_3}^2}{\gamma}. \tag{53}$$

With this condition the operation of the apparatus becomes completely self-contained.

5 Overall Operation

We can now summarise the overall behaviour of the apparatus in its quasi-steady burning stage.

We firstly pose the pressure, temperature and velocity changes across the filter simply in terms of the relations

$$p_3 = f_p p_2, \qquad T_3 = f_T T_2 \qquad \text{and} \qquad u_3 = (1-\alpha)\frac{f_T}{f_p}u_2 \tag{54}$$

where the functions $f_p(t)$ and $f_T(t)$ would need to be obtained by solving an adequate model for the filter or by experimental evaluation. Moreover, assuming that all solid particles are trapped by the filter, one has that

$$\rho_3 u_3 = (1-\mu)\dot{m}/A_3 \qquad \text{with} \qquad p_3 = R\rho_3 T_3. \tag{55}$$

It seems likely that both $f_p(t)$ and $f_T(t)$ would assume values slightly less than unity under normal operation. That is, relatively small pressure and temperature losses would appear across the filter.

Based on simple mass and pressure balances, but to a good degree of accuracy, the steady-state behaviour of the combustion chamber gives

$$p_2 = \mathcal{P}, \qquad T_2 = T_B \qquad \text{and} \qquad \rho_2 u_2 = \dot{m}(\mathcal{P}). \tag{56}$$

Applying the choking condition (53), together with these results, one finds that the pressure in the combustion chamber is given by the algebraic relation

$$\frac{\mathcal{P}^2/RT_B}{\dot{m}^2(\mathcal{P})} = \frac{\mathcal{P}^{2-b}}{aRT_B} = \frac{f_T}{f_p}\frac{(1-\mu)^2}{\gamma A_4^2}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} . \tag{57}$$

This relationship applies provided the value of A_4 is small enough to ensure choking. Noting the result (51), the condition for this to occur is that

$$A_4^2 \leq \frac{\dot{m}^2(p_\infty/f_p)}{p_\infty^2/RT_B} f_p f_T \frac{(1-\mu)^2}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} . \tag{58}$$

In general, the rate of production of gases to fill the airbag is increased as the nozzle area A_4 is decreased below this value.

6 Conclusions

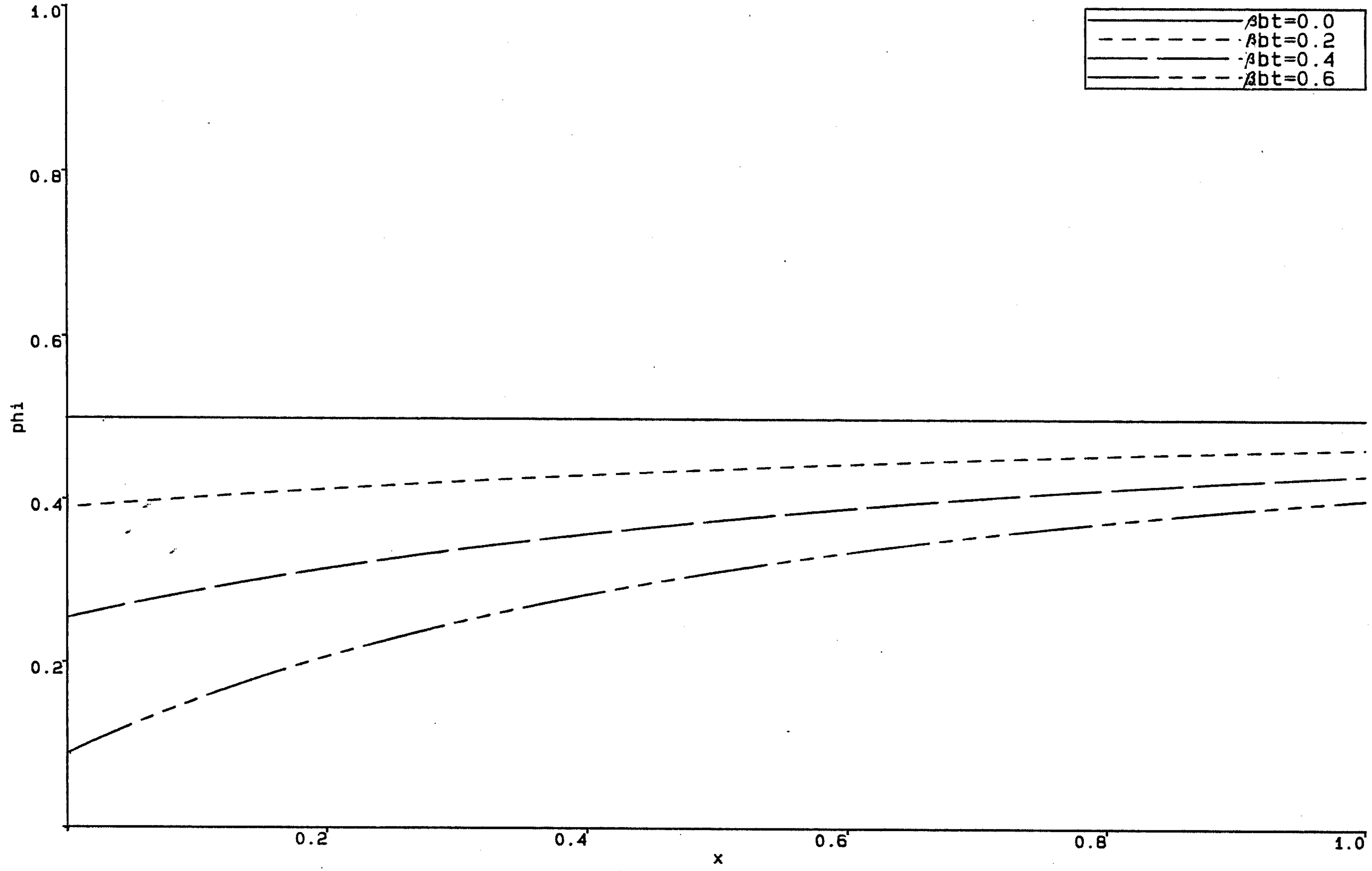
Various aspects of the behaviour of the combustion chamber have been discussed on physical and semi-empirical grounds. The most salient feature is that an almost uniform pressure would tend to be found under normal burning conditions. The examination could be extended using numerical solutions of less simplified models.

The filter model has enabled us to predict the pressure drop in the filter and the deposition of particles and consequent 'clogging up' of the filter. To give quantitative results, estimates for the filter parameters, A, β and K are needed. The neglect of heat transfer in the filter is probably not realistic and some thought is needed as to how to incorporate heat transfer. We assumed that the filter has uniform porosity initially; if $\varphi = \varphi_0(x)$ at $t = 0$ then the analysis is similar but slightly more complicated since the solution for φ must in general be left in integral form. We assumed that the particles become immobile on collision with the filter; in practice they may bounce and interact with the gas.

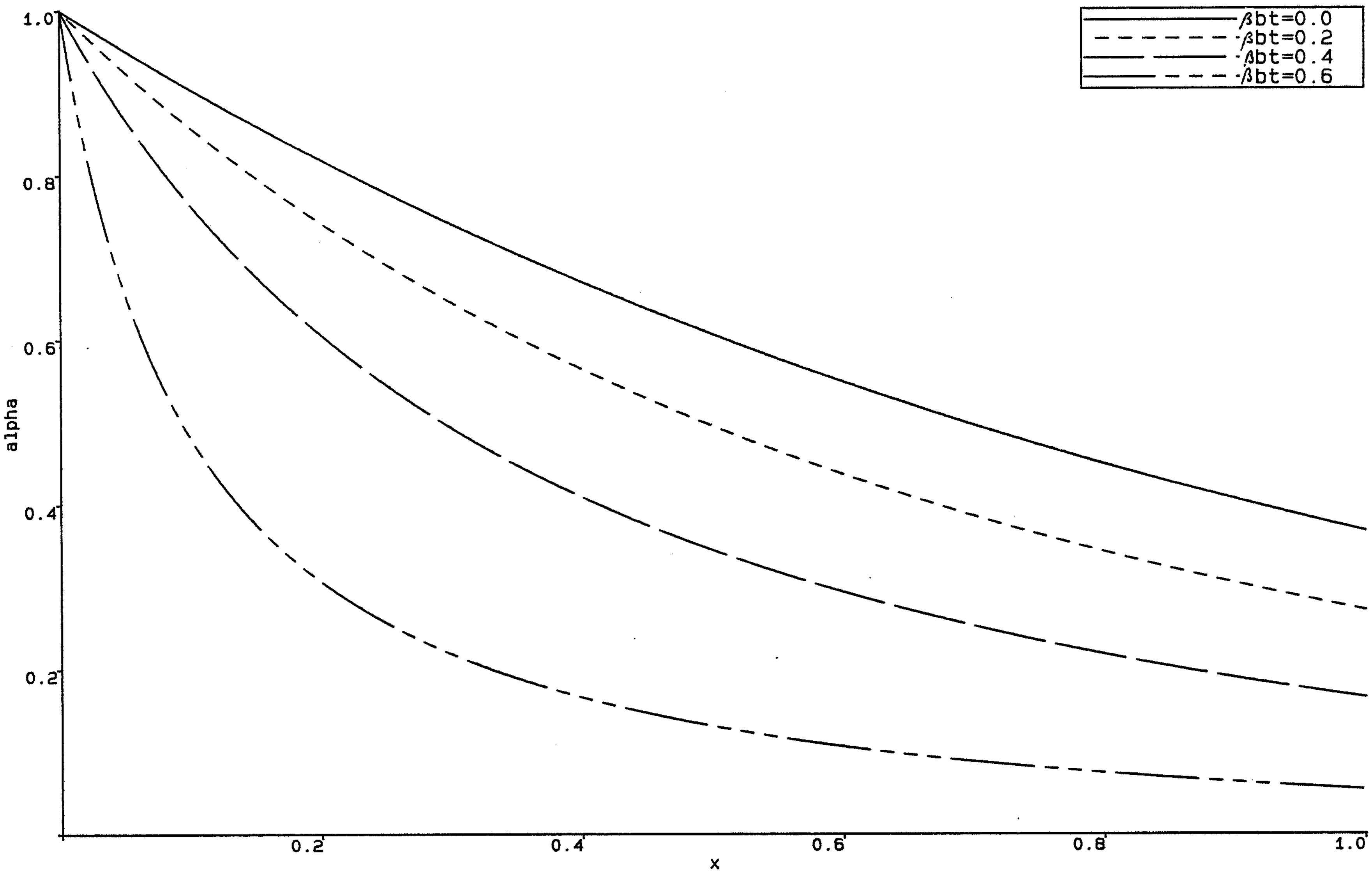
The model for the nozzle is straightforward and it would seem that a choked condition at the outlet would be the most desirable operating condition to ensure that high pressures are maintained in the combustion chamber. With the appropriate choking or non-choking conditions applied, the overall behaviour of the apparatus is reasonably well predicted.

JND	SJC	ELT
CA	LH	JWD
DSR	SDH	JL
NS	PW	ADF

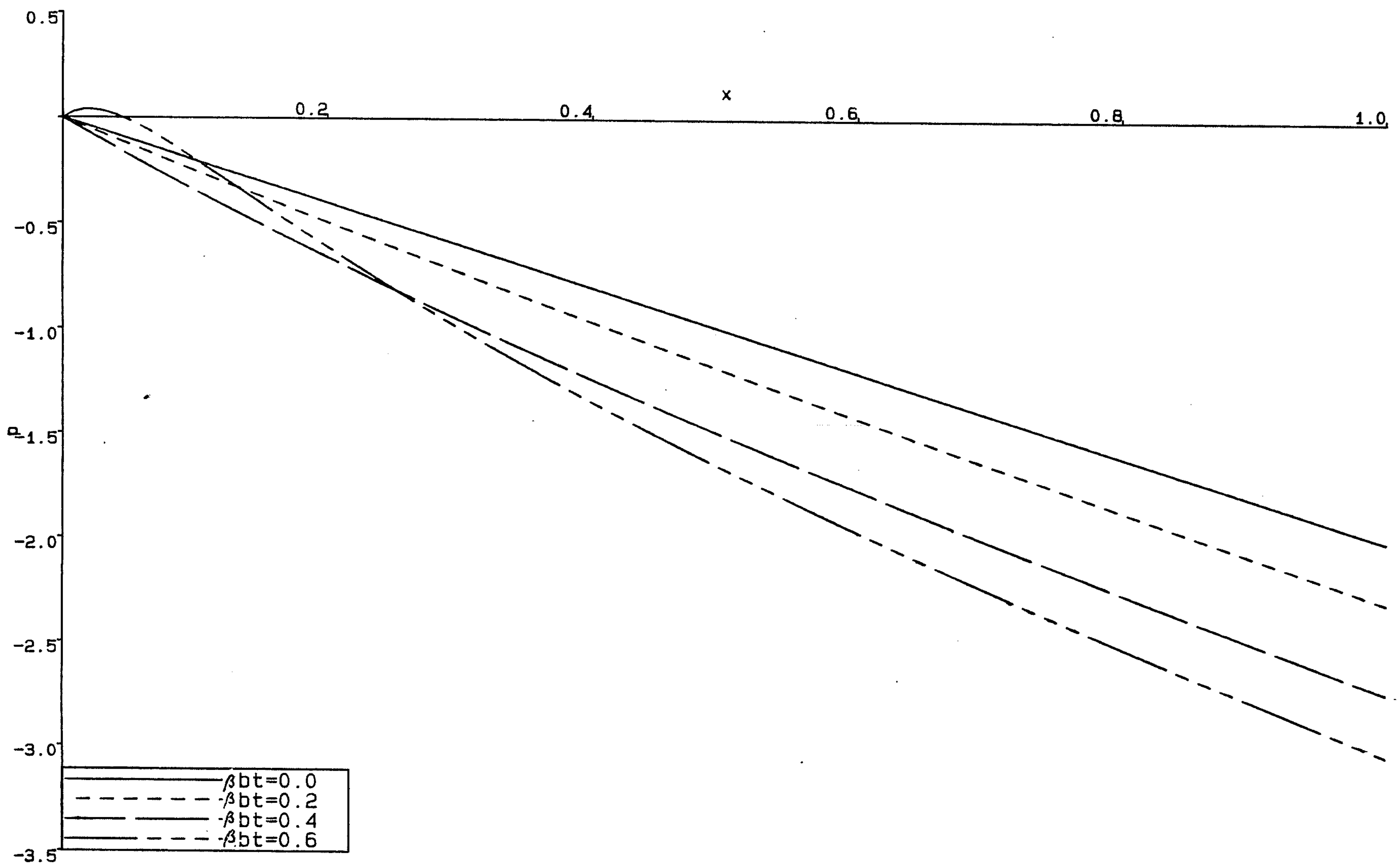
(Report by ELT, CA, JWD and DSR)



Porosity against distance, K=2.0



Solid volume fraction, K=2.0, a=1



Gas pressure, K=2.0, g=1.0, A=15.0