## Electric arc problem

Introduction

This problem was concerned with the behaviour of an electric arc as the current supplied to the arc passed through zero. It was prompted by a series of experiments by S.Smith showing that under some but not all circumstances the arc was extinguished and we sought to establish a model predicting which form of behaviour would occur.

The basic geometry of the arc is
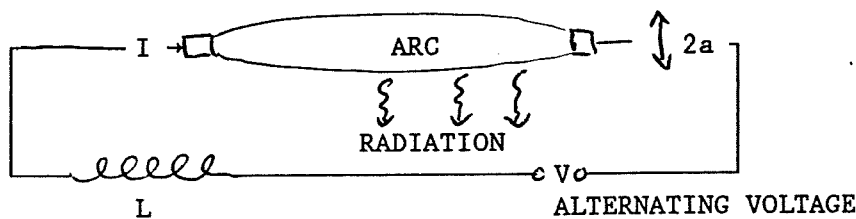
and it comprises a region of hot gas which is at a sufficiently high temperature to conduct electricity well. Heat is generated and lost by radiation and conduction. We were mainly concerned with the behaviour of optically thin arcs with a radius of around 100μm.

There is a substantial literature on the behaviour of electric arcs but much of it is unsatisfactory as it requires various ad-hoc assumptions to produce a complete equation, for example Steenbeck's principle.

During the Study Group we attempted to show how an equation of state could be derived by proposing that the arc should be in thermal equilibrium with the surrounding gas. This led to a closed set of equations which could be solved.

We looked at the behaviour of this system as $I \to 0$ but were unable to draw any firm conclusions for in the parameter range considered the time scales for the arc to cool and the electrical circuit to respond are almost identical. Hence the arc may (or may not) cool down sufficiently to extinguish.

This report is concluded with a brief statement of a more complete model of the arc studied in the Study Group which should hopefully lead to some future research.

1. Derivation of the basic model

We consider an arc to be a region of hot gas at temperaure T. The arc itself is approximately cylindrical.

The conductivity of the air is $\sigma(T)$ and an electric field E is set up within it by an external circuit. Within the arc we have
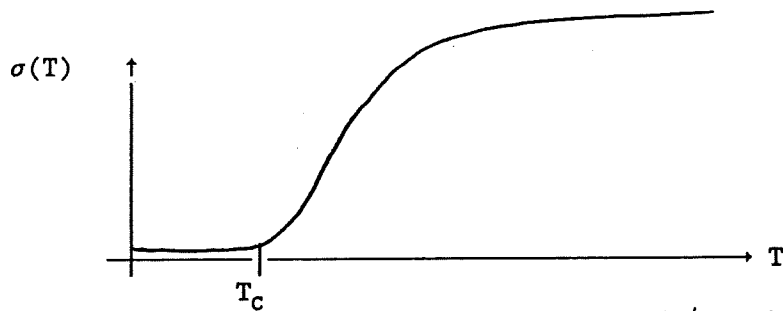
(i)   Heating due to current flow equal to $\sigma(T)E^2$

(ii)   A conduction term $k\nabla^2 T$ where $k \approx 10^{-2}$.

(iii) Heat loss due to radiation which for an optically thin arc is

$$I_{RL} = 2K\Lambda T^4$$

where $2K\Lambda \approx 10^{-3}$.

The conductivity $\sigma(T)$ has the explicit form

$$\sigma(T) \approx 1000T^{3/4}\exp(-6.53\text{x}10^4/T)$$

We consider this to be negligible if $T < T_c \approx 2\text{x}10^4$. There is also a term due to heat gain by radiation within the arc which takes the form

$$I_{RG} = \int_{arc} T^4 G(r,r')dr' = \text{gain at point r where } G(r,r') \text{ depends on the}$$

geometry of the arc.

Hence the heat balance equation is

$$\rho Cp \frac{\partial T}{\partial t} = k\nabla^2 T + \sigma(T)E^2 - 2K\Lambda T^4 + I_{RG}.$$

As an approximation we shall consider the field within the arc to be a constant independent of r. (This seems reasonable for a long thin arc but may be worth calculating more exactly in later work). In this case if the arc radius is a(x) at the point x then from Ohm's law

$$I = 2\pi E \int_0^a \sigma(T) r \, dr. \tag{2}$$

To determine the arc we need to find T, E and a. So far we have only two equations. The approach in some of the literature is to treat T as a constant independent of r and to ignore I_{RG}; thus from (1),(2),

$$\left. \begin{array}{l} I = \pi a^2 E \sigma(T) \\[2mm] \text{and} \quad \sigma(T) E^2 = 2 K \Lambda T^4 \end{array} \right\} \tag{3}$$

Steenbeck's principle is to take the <u>minumum E satisfying (3)</u>. Thus

$$\frac{d}{dT} E = 0 \quad \text{so} \quad \frac{d}{dT} (T^4 / \sigma(T)) = 0 \tag{4}$$

$$\text{or} \quad \frac{d}{dT} [\log \sigma(T)] = 4/T,$$

and (4) then provides a third equation of state. Not only is this ad-hoc it also (as Steve Smith showed) leads to various contradictions. We now propose an alternative mechanism for closing the system.

2. Closure of the system on the assumption of thermal equilibrium

When we examine the form of $\sigma(T)$ we see that for T less than a value T_c then $\sigma(T)$ is effectively zero. We thus can <u>define</u> the arc to be the region of gas for which

$$T > T_c, \quad \text{so} \quad T(a) = T_c, \quad a: \text{radius of arc}.$$

If $T < T_c$ then no current flows and hence the arc is <u>not</u> ohmically heated.

We now consider the gas surrounding the arc. This will receive heat from the arc through conduction and radiation and will also radiate heat itself.

Hence the equation for the temperature in the gas external to the arc

is

$$\rho C_p T_t = k\nabla^2 T + I_{RG} - I_{RL}$$

Radiant heat $\nearrow$ $\nwarrow$ Radiant heat
gained (global   lost (local term)
          integral)

To make progress we shall make the gross assumption that I_{RG} and I_{RL} balance
in the gas   [Clearly this will in general be a crude approximation: we
should think more clearly about approximating the global integral terms], so
that outside the arc

$$\rho C_p T_t = k \nabla^2 T. \tag{5}$$

3. Solution of (2),(4) in the steady state

To solve (2), (4) we make the assumption that the arc is long and thin
and slowly varying in x. Hence we may examine its behaviour by using slender
body theory. We firstly look at the gas outside the arc.

Let the length of the arc be ℓ; then we can approximate it to be a line
of sources. Hence the solution of (5) is

$$T(r,x) = \int_0^\ell f(x')/\sqrt{(x-x')^2 + r^2} \, dx' \tag{6}$$

Similarly, on the assumption of slow variation   we can write the solution of
(4) close to arc as

$$T = A(x) + 2B(x) \log(1/r') \tag{7}$$

where $r = \varepsilon r'$ and $\varepsilon \ll 1$ is a measure of the aspect ratio of the arc. From (5)

$$T = \int_0^\ell \frac{f(z')dz'}{\sqrt{(z-z')^2 + \varepsilon^2 r'^2}}$$

$$= \int_0^\ell f(x)/\sqrt{(x-x')^2 + \varepsilon^2 r'^2} + \int_0^\ell \frac{f(x')-f(x)}{\sqrt{(\dots)}}$$

$$= 2f(x) \log(1/\varepsilon r')$$

$$+ f(x)\log(4x(\ell-x))$$

$$+ \int_0^\ell [f(x')-f(x)]/\sqrt{(x-x')^2 + \varepsilon^2 r'^2} \tag{7}$$

Comparing (8) with (7) and matching to leading order we see that

$$f(x) = B(x) \text{ and}$$

$$A(x) = 2B(x)\log(1/\varepsilon) + B(x)\log(4x(\ell-x)) + \int_0^\ell \cdots$$

Now when $r = a$, $T = T_c$. But $a = \varepsilon a'$ and $a(x)$ is an order one quantity. Hence

$$T_c = A(x) + 2B(x) \log(1/a'),$$

therefore

$$T_c - 2B(x) \log(1/a') = 2B(x) \log 1/\varepsilon$$
$$+ B(x) \log 4x(\ell-x)$$
$$+ \int_0^\ell \frac{B(x)-B(x')}{|x-x'|} dx'.$$

Thus to leading order $B(x) = T_c/[2\log(1/\varepsilon)]$ which is a constant. To next order

$$B(x) = T_c/\log[(1/\varepsilon a')^2 . 4x(\ell-x)],$$

thus

$$B(x) = T_c/\log[1/a^2 4x(1-x)].$$

We can repeat this procedure to refine our estimates.

Now $T(r) = A(x) + B(x) \log(\varepsilon/r)$, so $\left.\dfrac{dT}{dr}\right|_{r=a} = - B(x)/a$,

and we can conclude that on the boundary of the arc

I) $T(a) = T_c$, II) $\dfrac{dT}{dr}(a) = - T_c/\left(a \log\left[\dfrac{4x(\ell-x)}{a^2}\right]\right)$.

These now become our equations of state. (But see Andrew Lacey's comments in the appendix on the problem when we do not treat heat transfer to the gas as purely conductive).

4. Solution of the steady state equations of the arc

The arc steady state itself satisfies the equations

$$K(T'' + \frac{1}{r} T') + \sigma E^2 = 2K\Lambda T^4 \qquad (9)$$

$$2\pi E \int_0^a \sigma(T) r dr = I \tag{10}$$

$$T'(0) = 0 \tag{11}$$

$$T(a) = T_c \tag{12}$$

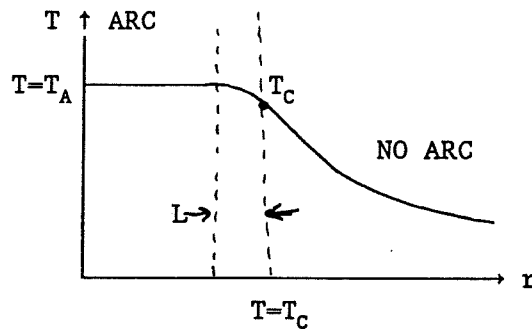$$T'(a) = - T_c /(a \log\left[\frac{4x(\ell-x)}{a^2}\right]) . \tag{13}$$

Here $k = 10^{-2}$ and $K\Lambda T^3 = 8 \times 10^9$ (SI units) so conduction becomes important on a length scale

$$L = 1/\sqrt{10^2 \times 8 \times 10^9} \approx 1\mu m.$$

Note $\quad a \approx 100\mu m$.

(a) The case $a \gg L$

The temperature in this case has the form

If $r < a - 0(L)$ then

$$T = T_A, \text{where} \quad \sigma(T_A)E^2 = 2K\Lambda T_A^4 \text{ and } \pi E a^2 \sigma(T_A) = I.$$
and

Now at the boundary of the arc

$$T'(r) \approx - T_c /(a \log\left[\frac{4x(\ell-x)}{a^2}\right])$$

and $- T'(a) \approx (T_A - T_c)/L.$

Hence $\quad T_A \approx T_c\left[1+(L/a) \log\left[\frac{4x(\ell-x)}{a^2}\right]\right] \tag{14}$

If we approximate $\sigma(T)$ the piecewise linear function

$\sigma(T_A) \approx \sigma'(T_c)(T_A - T_c)$ then $\sigma'(T_c) \approx 15$ so

$$\sigma(T_A) \approx 15 T_c L \, a \, \log\left[\frac{4x(\ell-x)}{a^2}\right].$$

We can now solve the system

$$a = (I^2/2K\Lambda T_A{}^4 \pi^2 \sigma(T_A))^{\star}$$
$$\approx (I^2/2K\Lambda T_A{}^5 15.\pi^2 L/a)^{\star}.$$

Substituting values we have

$$a \approx 1.3 \times 10^{-5} I^{2/3}. \tag{14}$$

Similarly

$$E \approx I/\pi a^2 \times \frac{a}{15 T_c L} \approx I/(15\pi T_c L a),$$

i.e. $\quad E \approx 8.16 \times 10^4 I^{1/3}. \tag{15}$

These formulae apply provided $a \gg L$.

We note that E and a decrease as a function of I, and T_A slowly

increases.

When $I = 0.0213$ (14) $\Rightarrow a = L$. If $I = 10^4 \sin 2\pi \times 50t$ this occurs within

7.10^{-9}s of the current zero.

(b) The case $a \ll L$. Rescale $r \to r/a$, so in the new variable

$$T'' + \frac{T'}{r} + \frac{a^2}{K}[\sigma E^2 - 2\Lambda K T^4] = 0.$$

From (13)

$$T'(1) = T_c/\log[4z(\ell-z)/a^2]$$

If a is small then $T^4 a^2$ is small hence

$$T'' + \frac{T''}{r} + \frac{a^2}{K}\sigma E^2 = 0$$

(This is called the Elenbaas-Heller equation). But if $\sigma(T) = 15(T-T_c)$, putting
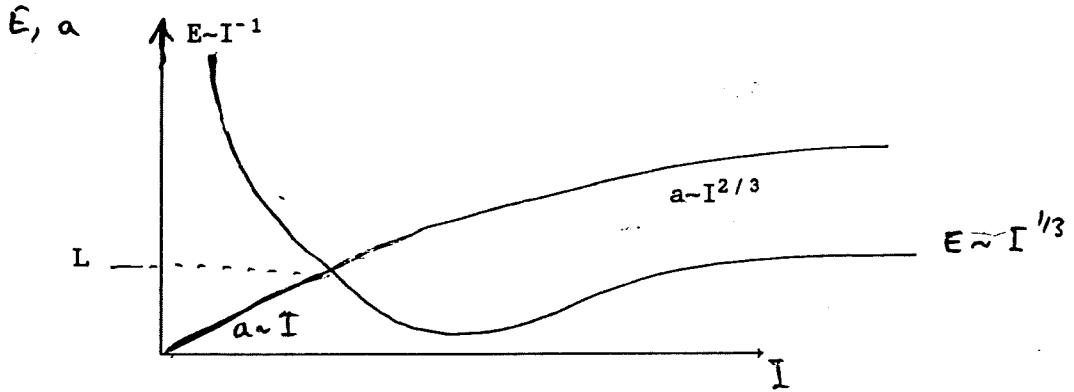
$$u = T - T_c \text{ gives}$$

$$u = A J_0\left(\sqrt{\frac{15}{K}} \, aEr\right)$$

Now $\quad T(1) = T_c$, $u(1) = 0$, therefore $J_0\left(\sqrt{\frac{15}{K}} \, aE\right) = 0$. Hence $aE = \sqrt{K/15} \, j_0$

where j_0 is the first zero of J_0. Therefore

$E \sim 1/a$ and solving we get $a \sim I$. (and $T \rightarrow T_c$).

We can combine the estimates in the following figure:

Note This predicts $E \rightarrow \infty$ as $I \rightarrow 0$. All heat loss is conductive with little

radiative heat transport. Of course the external circuitry cannot support such

large fields.

What we have described is a quasi-steady state in which thermodynamic

equilibrium is achieved quickly. We now consider the fully time dependent

case.

5. Time-dependent arc

There are various timescales to be considered.

1. Radiative cooling

$$t_R \sim \rho C_p / 2K \Lambda T^3 \sim 10^{-7} s$$

2. Conductive cooling

$$t_c \sim \rho C_p a^2$$
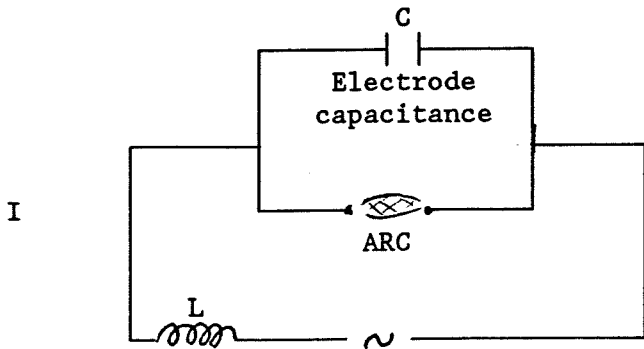
Only important if $a \approx L$, i.e. $t < 10^{-8} s$ (from before).

3. External circuitry

$$t_x = \sqrt{LC} \sim \sqrt{1 \times 10^{-3} \times 10 \times 10^{-12}} \sim 10^{-7} s$$

Thus the time for the external circuit to react is comparable to the

time it takes for the arc to cool down radiatively.

The complete system we need to consider is

$$V = V_0 \sin (2\pi \times 50t)$$

where the arc forms part of the system. To maintain thermal equilibrium our earlier equations imply that $E \sim I^{1/3}$ for I away from zero and I^{-1} for smaller I. As the external circuitry responds in time $O(10^{-7})$s and E must change sign over a half cycle it is clear that in the 10^{-7}s during which E decreases to zero the ohmic heating in the arc cannot balance the heat loss and it will start to cool. If the arc cools below the temperature $T = T_c$ it will not restart on the next half cycle as now the gas has effectively an infinite resistance.

It is clear though that the cooling time and the circuitry response time are so close that we cannot decide this question with the approximations made during the Study Group.

6. A review of the complete model

It is worth finally stating a complete model.

We shall take the arc to be the region of hot gas between two electrodes.
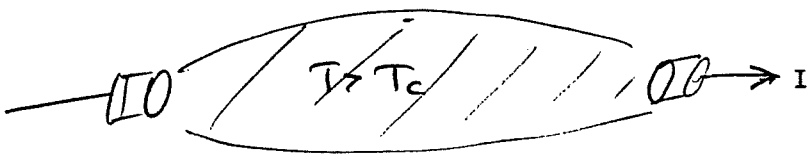
We solve the following system:

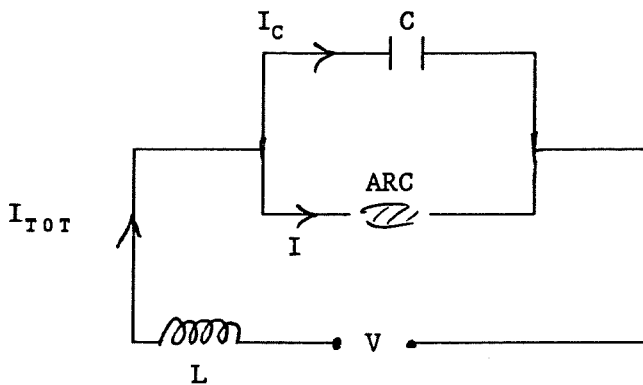A. Inside the arc. If we continue to use a uniform field assumption

(1) $\rho C_p \, T_t = \nabla \cdot (k(T) \nabla T) + \sigma(T) E^2$

$$- 2K\Lambda T^4 + \int_{\text{space}} 2K\Lambda T^4 G(r, r') dr$$

(where we must find G); $T \to T_{\text{AMBIENT}}$ at ∞ .

(2) $I = 2\pi E \int_0^\infty \sigma(T) r \, dr$

(1) Holds everywhere but $\sigma(T)$ terms only matter if $T > T_c$.

B. Outside the arc.

I_c: Capacitative current due to changes on the electrodes.

1. $I_{TOT} = I + I_c$ (current conservation)

2. $V = L\dot{I}_{TOT} + E\ell$ (Kirchoff)
                   (length of arc)

3. $E\ell = \int I_c / C$

Combining, we find

$$V = L\dot{I} + LC\ddot{E} + E\ell$$

Our initial attack should still assume that the hot part of the arc is cylindrical, although the global radiation term will make things harder.

CJB, AAL, SJC, JC, PWW, JRO

Appendix: written after the follow up meeting in Oxford in June.

ERDC Problem: Electric Arc in Bitumen

We wish to find the behaviour of

$$\rho c T_t = \nabla \cdot (k \nabla T) + E^2 \sigma(T) - 2 K \Lambda T^4 \qquad (1)$$

subject to $T \to T_0$ at some (relatively distant) boundary and $\iint \sigma E \, dx \, dy = I$ where the integral is taken over some typical cross-section of the arc. The terms in (1) represent rate of change of internal energy, heat diffusion, ohmic (Joule) heating, and heat loss by radiation, respectively. We may be thinking of I as either imposed or given by a circuit equation such as

$$V(t) = \ell E + L \frac{d}{dt} (I + \ell c \frac{dE}{dt}) + R(I + \ell c \frac{dE}{dt}) . \qquad (2)$$

where $V(t)$ is a known applied voltage. The conductivity σ is given by

$$\sigma(T) = A T^{3/4} e^{-\Phi/T}.$$

so we choose to scale temperature with Φ: $T = \Phi u$. Since T_0 is expected to be small compared with Φ the boundary condition becomes $u = 0$. We balance the source and sink terms by scaling E appropriately, we scale distance with a typical arc radius (10^{-4}m?), and we should scale time to suit this distance (rather than the period of $V = 2.10^{-2}$s).

Taking k to be constant (we could probably modify the analysis to cope with $k = k(T)$) we get:

$$\left. \begin{array}{l} \alpha u_t = \delta^2 \nabla^2 u + E^2 \sigma(u) - u^4 , \\[2mm] u = 0 \text{ for large } r \\[2mm] I = 2 \pi E \int r \sigma(u) \, dr \end{array} \right\} \qquad (3)$$

Notes: We have assumed an optically thin arc. If it is actually optically thick the $-u^4$ term is replaced by a modified diffusion term. Since we are away from equilibrium the radiating term should probably be a function of u other than u^4; the following analysis can easily be modified to allow for this.

We expect α and δ both to be small. The former leads us to neglect the time derivative to give us a quasi-steady problem while the latter leads to the appearance of a boundary layer, say at $r = a$ (the surface of the arc), which is the only place that spatial derivatives are significant.

We have:

$r < a$ (arc interior) $E^2\sigma(u) \sim u^4$;   (4)

$r > a$ (exterior) $u \sim 0$;

$r = a + \delta x$ (arc surface) $u'' + E^2\sigma(u) - u^4 \sim 0 \ (' = \frac{d}{dx})$.   (5)

The current condition then gives $\pi a^2 E\sigma(U) = I$ where $u = U$ solves (4), i.e. $U^4/\sigma(U) = E^2$.

We now have two equations for the electric field E, internal arc temperature U, and arc radius a. The final equation is to come from careful consideration of the boundary layer at $r = a$ subject to matching with $r < a$ and $r > a$ ($u \to U$ and $u \to 0$ respectively).

Multiplying (5) by u' and integrating from $x = -\infty$ ($u=U, u'=0$) we see that

$$\int_0^U (E^2\sigma(u) - u^4)\,du = 0 \tag{6}$$

or

$$U^4 \int_0^u \sigma(u)\,du/\sigma(U) = \frac{1}{5} U^5$$

i.e.

$$\int_0^1 \frac{\sigma(Us)}{\sigma(U)}\,ds = \frac{1}{5}.$$

Taking the given conductivity law $\sigma(u) - u^{3/4}e^{-1/u}$ gives

$$F(U) = \int_0^1 s^{3/4} \exp \left\{ \frac{-(1-s)}{Us} \right\} ds - \frac{1}{5} . \tag{7}$$

It is clear that $F(U)$ is increasing with $F(U) \to 0$ as $U \to 0$ and $F(U) \to \frac{4}{7}$ as $U \to \infty$. We see that there is a unique solution $U = U^*$ to (7). To compare this predicted value with that given by Steenbeck's minimum principle $(0 = \frac{d}{du}(\frac{\sigma}{u^4}) = u^{-6}e^{-/u}(1-13u/4)$, i.e. $U = 4/13)$ we look at the integrand in (6).

Since $u^4/\sigma(u)$ is positive and large for positive u both large and small, and $\frac{d}{du}(u^4/\sigma/u) < 0$ for $u < \frac{4}{13}$, > 0 for $u > 4/13$ it is clear that $u^4 - E^2\sigma(u)$ is positive for all $u > 0$ if $E^2 < \xi_0 = (4/13)^4/\sigma(4/13)$, is positive for $0 < u < 4/13$ and for $u > 4/13$ but zero for $u = 4/13$ if $E^2 = \xi_0$, is positive for $u < u < u_1$ and for $u > u_2$ but negative for $u_1 < u < u_2$ if $E^2 > \xi$, where u_1 and u_2 are respectively decreasing and increasing functions of E^2 with $u_1(\xi_0) = u_2(\xi_0) = \frac{4}{13}$.

$u_t(E^2)$

$E^2 < \xi_0$

$E^2 = \xi_0$

$E^2 > \xi_0$

$u_1 \qquad 4/13 \qquad u_2 \qquad u$
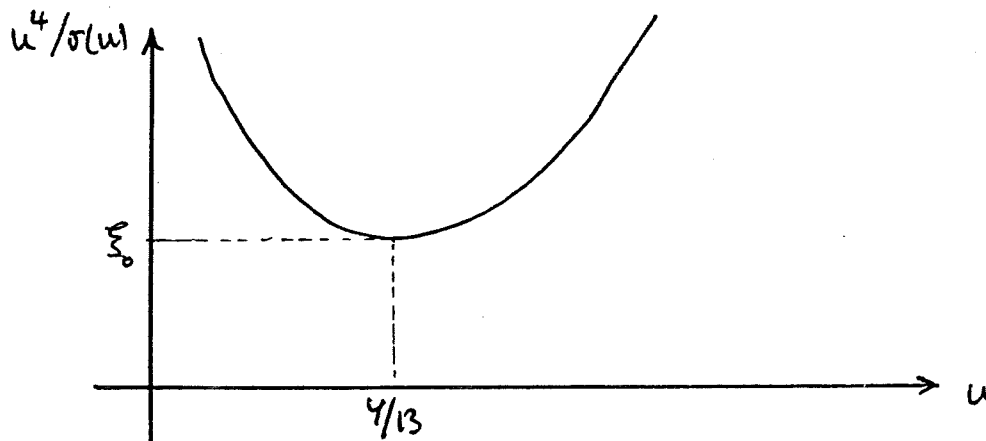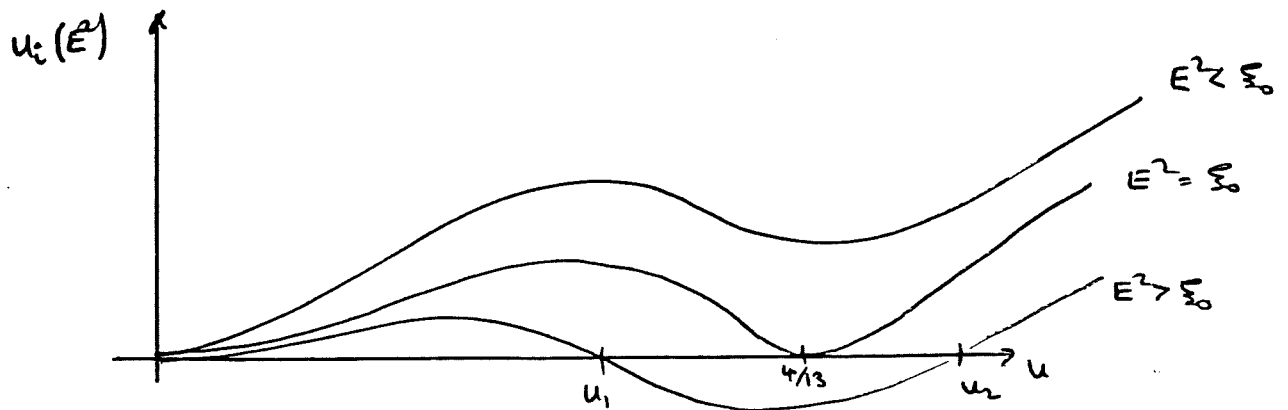
We see that $U = u_2(E^2)$ with $\int_0^u (u^4 - E^2 \sigma(u))du \ast 0$. Thus $E^2 > \xi_0$ and

$U = u_2(E^2) > 4/13$.

Improved estimates can be found as asymptotic series e.g.

$U \sim U^* + U' + \ldots$, with U_1 (and the first order corrections for other

equations) depending upon a power of δ and of $\ln \frac{1}{\delta}$ but these do not seem very

illuminating.

To see how the solution becomes invalid, rather than use detailed

asymptotics we may now find how the αu_t or $\delta^2 \nabla^2 u$ terms become important. Just

how this happens depends critically upon the sizes of α and δ. The diffusion

term will become significant throughout the arc for $a = 0(\delta)$ while the

internal energy change must be considered in the boundary layer if

$\frac{da}{dt} = 0(\delta/\alpha)$. Taking I to vary linearly with time so that $a = a_1\sqrt{-t}$ we must

have diffusion everywhere for $- t = 0(\delta^2)$ and initial energy at the arc

boundary for $- t = 0(\alpha^2/\delta^2)$ (if $\alpha \geq 0(\delta)$ the above boundary layer analysis

would be incorrect).

For $\delta^2 >> \alpha$ (which we believe to be the case) the diffusion term

becomes significant first and we must then consider the (quasi-steady)

problem

$$\nabla^2 u + E^2 \sigma(u) - u^4 = 0,$$

$$u = 0 \qquad r \text{ large},$$

$$I = 2\pi E \int r\sigma(u)dr,$$

where r has been rescaled.

Since we are looking for two-dimensional, radially-symmetric solutions the problem becomes

$$u" + u'/r = f(u) \quad r > 0, \text{ where } f(u;E) = u^4 - E^2\sigma(u),$$

with $u'(0) = 0$, $u(r) \to$ as $r \to \infty$.

Multiplying by u' and integrating from 0 to ∞ we get

$$0 < \int_0^\infty \frac{u'^2}{r} dr = - \int_0^u f(u)du, \text{ where now } U = u(0).$$

This indicates that $E^2 > U^* 4/\sigma(U^*)$ and also $u_1(E^2) < U < u_2(E^2)$. It appears that I decreases as E^2 increases; it is not immediately clear how the maximum temperature U behaves.

It should be noted that for short times the circuit equation may have to be considered more carefully. Again we must be sure of the relevant sizes to ascertain how the external problem links with the arc. We also note that for $\delta^2 < < \alpha \leq 0(\delta)$ we can adapt the earlier boundary layer analysis to include a term of the type $Vu'(V = - \frac{da}{dt})$, coming from αu_t, which gives some relation between V and U. Preliminary analysis (which can be taken further if it is at all relevant) suggests that $V = V(U)$ is a decreasing function for $U > \frac{4}{13}$ but for $U \leq \frac{4}{13}$ we only know (at present) that $V \leq V_0(U)$ for some decreasing function V_0 satisfying $V_0(\frac{4}{13}) = V(\frac{4}{13}+)$. While $U > \frac{4}{13}$ the equation (4) gives E^2 decreasing with U.

A.A.L.