Circular arc spline approximation of pointwise curves for use in NC programing

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1. Introduction

We consider a numerical control (NC) cutting machine which can cut only line segments and circular arcs. Thermal cutting processes require constant tool velocity because

- too slow velocity leads to overheating and melting,
- too fast velocity interrupts the cutting process.

The inputs with which the machine works are sets of points in a particular order which are in Cartesian plane.
From a set of points (inputs) we must create a sequence of line segments and circular arcs that pass through some of the points and are "sufficiently close" to the others – ϵ error condition. The case in which the points can be approximated with straight line segments is well investigated. We are interested in the sets of points which can only be approximated by arcs. Below we formulate this particular task.

2. The problem

A sequence of N points is given. A curve must be created, composed of circular arcs, such that:

- it passes through/nearby the given points in the same sequence;
- the Hausdorff distance between the points and the curve does not exceed a certain value ϵ;
- it is composed of minimal number of arcs;
- the output should consist of sets of the type:
  $\{(x_1, y_1), (x_2, y_2), (x_c, y_c), E\}$,

where (x_1, y_1) and (x_2, y_2) are respectively the initial and the final points of a certain arc, (x_c, y_c) is its center and $E = +1$ if the direction of the arc is counter clockwise or $E = -1$ if the direction of the arc is clockwise.

Remark. Local minimum – fitting an arc to each set of 3 points – is not a solution of the task.

2.1. Summary of the approach

- We begin with a program for finding the center and the radius of a circle that passes through three fixed points.
- Having such a program we make another one for finding the "best" arc that connects two fixed points (which have at least two inner points between them). This arc passes through the two fixed points and through one of the points between them.
- Next we find the "best" arc between any two points (that have at least two inner points) of the set of points we are given.
- From the set of arcs that we have created, we exclude those that do not satisfy our error condition.
- From the arcs that are left we may choose different ways to get from the initial point to the last. We chose such a path that contains minimal number of arcs. Usually the connecting points are spread almost uniformly throughout the set we are given.

2.2. An arc through three fixed points

Let us have the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, Fig. 1. The midpoints A and B of the line segments connecting (x_1, y_1) and (x_2, y_2) and (x_2, y_2) and (x_3, y_3) have coordinates (x_A, y_A), (x_B, y_B). Obviously

$$x_A = \frac{x_2 + x_1}{2} \ , \ x_B = \frac{x_3 + x_2}{2}$$

and

$$y_A = \frac{y_2 + y_1}{2} \ , \ y_B = \frac{y_3 + y_2}{2}.$$

The equations of the lines that pass through the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ are respectively

$$l_1 : A_1 x + B_1 y + C_1 = 0$$

and

$$l_2 : A_2 x + B_2 y + C_2 = 0,$$

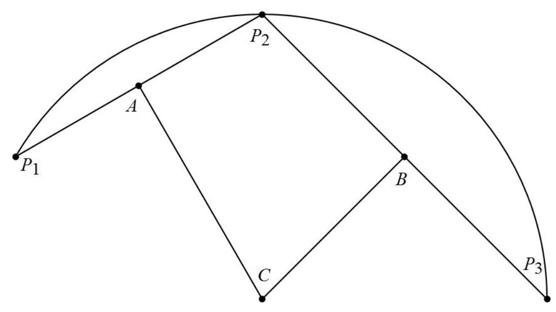

Figure 1: The center C of the circle through P_1, P_2, P_3

where $A_1 = y_2 - y_1$, $B_1 = x_2 - x_1$, $C_1 = -x_1(y_2 - y_1) + y_1(x_2 - x_1)$, $A_2 = y_3 - y_2$, $B_2 = x_3 - x_2$, $C_2 = -x_2(y_3 - y_2) + y_2(x_3 - x_2)$. Now, since the vectors $p_1(A_1, B_1)$ and $p_2(A_2, B_2)$ are orthogonal respectively to the lines l_1 and l_2 and we have the coordinates of A and B, we can easily find the equations of the line bisectors of the arcs that are orthogonal to l_1 and l_2 and pass respectively through (x_A, y_A) and (x_B, y_B). We have

$$b_1 : B_1x - A_1y + (-B_1x_A + A_1y_A) = 0,$$

$$b_2 : B_2x - A_2y + (-B_2x_B + A_2y_B) = 0.$$

The center $C(p, q)$ of the circle is where the two line bisectors intersect. Its coordinates are the solution of the system

$$B_1x - A_1y + (-B_1x_A + A_1y_A) = 0,$$

$$B_2x - A_2y + (-B_2x_B + A_2y_B) = 0.$$

So we have that

$$p = -\frac{-A_2B_1x_A + A_1B_2x_B + A_1A_2y_A - A_1A_2y_B}{A_2B_1 - A_1B_2},$$

$$q = -\frac{-B_1B_2x_A + B_1B_2x_B + A_1B_2y_A - A_2B_1y_B}{A_2B_1 - A_1B_2}$$

As for the radius of the circle, it is equal to the distance between the center and any point on it. We can use the point $P_1(x_1, y_1)$. We have that

$$r = \sqrt{(x_1 - p)^2 + (y_1 - q)^2}.$$

The direction of the arc is positive (negative) exactly when the orientation of the triangle $\overrightarrow{P_1P_2P_3}$ is positive (negative). This orientation is equal to the sign of the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{vmatrix}.$$

2.3. "Best" arc

Let us consider the task for connecting two fixed points $P_0(x_0, y_0)$ and $P_{n+1}(x_{n+1}, y_{n+1})$ (which have n inner points, $n \geq 2$) of our input set. First we build all the arcs that connect the two end points and pass through an inner one - that makes n arcs. Let r_i and $C_i(p_i, q_i)$, $i = 1, \ldots, n$ be respectively the radii and the centers of these arcs. For every arc with a center (p_i, q_i) and radius r_i, $(i = 1, \ldots, n)$ we calculate its Hausdorff distance to the inner points P_j, $j = 1, \ldots, n$.

$$d_{i,j} = \left| \sqrt{(x_j - p_i)^2 + (y_j - q_i)^2} - r_i \right|.$$

We now denote

$$d_i := \max\{d_{i,1}, \ldots, d_{i,n}\}.$$

For the i-th arc d_i is its greatest Hausdorff distance to an inner point. We remind that we now consider all the arcs that connect two fixed points and pass through a third between them. For the "best" arc of such kind we chose the k-th arc for which

$$d_k = \min\{d_1, \ldots, d_n\}.$$

"Best" arc – new suggestions.

The input set is the same: two fixed points $P_0(x_0, y_0)$ and $P_{n+1}(x_{n+1}, y_{n+1})$ (which have n inner points, $n \geq 2$). The midpoint M of the segment P_0P_{n+1} has coordinates (x_M, y_M). Obviously

$$x_M = \frac{x_0 + x_{n+1}}{2}, \qquad y_M = \frac{y_0 + y_{n+1}}{2}.$$

The equations of the line that passes through the point M and is perpendicular to the segment P_0P_{n+1} are:

$$c : \begin{cases} x_C = x_M + d * y10/w \\ y_C = y_M + d * x01/w \end{cases}$$

where: $x01 = x_0 - x_{n+1}$, $\quad y10 = y_{n+1} - y_0$, $\quad w^2 = (x01)^2 + (y10)^2$.

For $i = 1, \ldots, n$ we calculate the oriented distance d_i from M to the C_i-center of the circle through the points P_0, P_i, P_{n+1}

$$d_i = \frac{((x_i - x_M)^2 + (y_i - y_M)^2 - w^2/4).w}{2((x_i - x_M).y10 + (y_i - y_M).x01)}, \qquad d = \frac{1}{n}\sum_{i=1}^{n} d_i.$$

Next we define the center C of the optimal arc: C is at distance d from M. The radius of the arc is $r = \sqrt{d^2 + w^2/4}$. We calculate the errors e_i for the points P_i. Note that $e_i = \sqrt{(x_i - x_M)^2 + (y_i - y_M)^2} - r$ is the Euclidean distance between P_i and the point Q_i, which lies on this circle and on the radius through the point P_i. At the same time e_i is the Hausdorff distance between P_i and the optimal arc. More precisely this is one-side Hausdorff distance from given points to the found arc.

2.4. Next stages

Now we consider all the combinations of two points from our input set that have at least two inner points. For all such pairs of points we take the best (according to one of the ways previously described) arc that connects them. Since not all these arcs are close enough to all of their inner points (for an example we can rarely connect the first and last point with only one arc) we exclude those for which the distance between them and their inner points (at least one of them) is more than ϵ. Now we have a set of suitable arcs.
We may consider the problem for constructing a curve (made of arcs) from the first to the last point as a question for finding a path in a graph. We consider each point of the input set as a node and the arcs (connecting some of them and satisfying the error condition) as ribs.

For construction of the adjacency matrix $A = (a_{ij})_{i=1,\ldots,N}^{j=0,\ldots,N}$ we first set A to have only zeros. For $i = 1, \ldots, N-3$ (N is the number of the input points) we consider the best arc (rib) connecting the i-th and the j-th points ($j = i + 3, \ldots, N$). If this arc satisfies the error condition we predefine $a_{ij} = 1$.
We compare different paths by the length of their shortest arc (according to the number of inner points). One approach is to find all the paths in the graph we have derived and then chose the one in which the shortest arc is as long as possible. However, we have adapted an algorithm for finding a path with smallest amount of ribs. Usually the nodes we get are spread uniformly.

3. Numerical experiments

We have applied our approach to real examples. On Figure 2 the black curve consists of 200 points, that lie on the parabolic curve $y = 300 - 200 * (1 - x/500)^2$ and the white inner segments are the arcs (6 is their number), approximate the points.

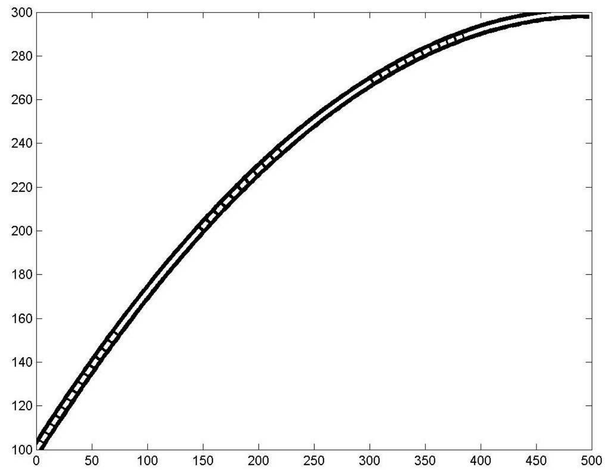

Figure 2: Approximation of the data by 6 arcs

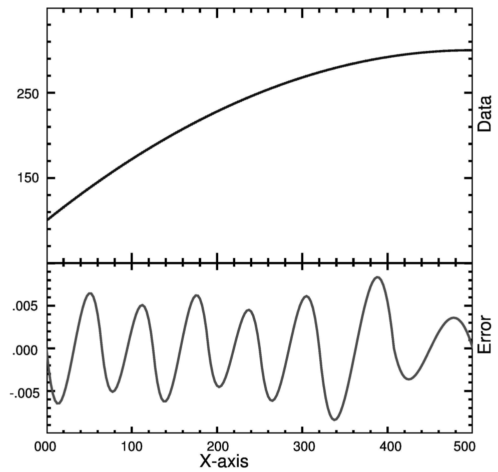

Figure 3: Approximation by 7 arcs (above) and the error of approximation (below)

On Figure 3 we show the approximation of the same data by 7 arcs and below we demonstrate how the error of approximation changes. The maximal error with 5 arcs is about 0.0183, but with 7 arcs – less than 0.0085. The output data for these two cases are:

Number of arcs is $N_{arc} = 5$

```
A (500.000,300.000) (370.000,286.480) (500.58361,  -337.37167)  1
A (370.000,286.480) (270.000,257.680) (514.73312,  -404.07667)  1
A (270.000,257.680) (177.500,216.795) (553.71037,  -509.27917)  1
A (177.500,216.795) ( 85.000,162.220) (628.01623,  -652.46917)  1
A ( 85.000,162.220) (  0.000,100.000) (744.77653,  -828.28417)  1
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Number of arcs is $N_{arc} = 7$

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A (500.000,300.000) (407.500,293.155) (500.20991,  -331.25917)  1
A (407.500,293.155) (320.000,274.080) (506.36069,  -370.56000)  1
A (320.000,274.080) (250.000,250.000) (525.32585,  -436.58167)  1
A (250.000,250.000) (190.000,223.120) (556.08512,  -513.63000)  1
A (190.000,223.120) (125.000,187.500) (602.69383,  -607.08750)  1
A (125.000,187.500) ( 65.000,148.620) (669.89912,  -719.13000)  1
A ( 65.000,148.620) (  0.000,100.000) (761.34933,  -850.08750)  1
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4. Summary

To recap, the problem was how to create a sequence of arcs

- passing through some of the given points and being sufficiently close to the others points,
- arcs must be as long as possible.

We did the following activities:

- examined the problem in the literature,
- developed an algorithm for constructing a sequence of arcs,
- tested our approach with a real data,
- improved the method,
- compared the results.

References

[1] Kazimierz Jakubczyk. Approximation of Smooth Planar Curves by Circular Arc Splines. May 30, 2010 (rev. January 28, 2012)

[2] O. Aichholzer, F. Aurenhammer, T. Hackl, B. Jüttler, M. Oberneder, and Z. Sír. Computational and structural advantages of circular boundary representation.