Chapter 4

A Seismic Inversion Problem for an Anisotropic, Inhomogeneous Medium

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4.1 Introduction

In the petroleum industry, many geoscientists rely on information provided by seismic data. Based on the interpretation of seismic data, geophysicists and geologists attempt to identify and target areas of economic importance, specifically hydrocarbon traps. We wish to extract from the data as much information as possible.

A key factor in the successful processing and interpretation of seismic data is knowing certain rock physics properties, in particular the elastic moduli of the section of earth through which seismic waves propagate. One means of attempting to quantify the rock physics properties is through a vertical seismic profile (VSP), wherein seismic sources are located on the surface and receivers are placed in a well-bore.

Traditionally, only traveltimes were measured at the receivers, but technological advances now allow measurement of the polarization angles (*i.e.*, particle displacement) as well. This provides an additional piece of information towards a practical means of accurately determining the physical parameters that characterize the medium.

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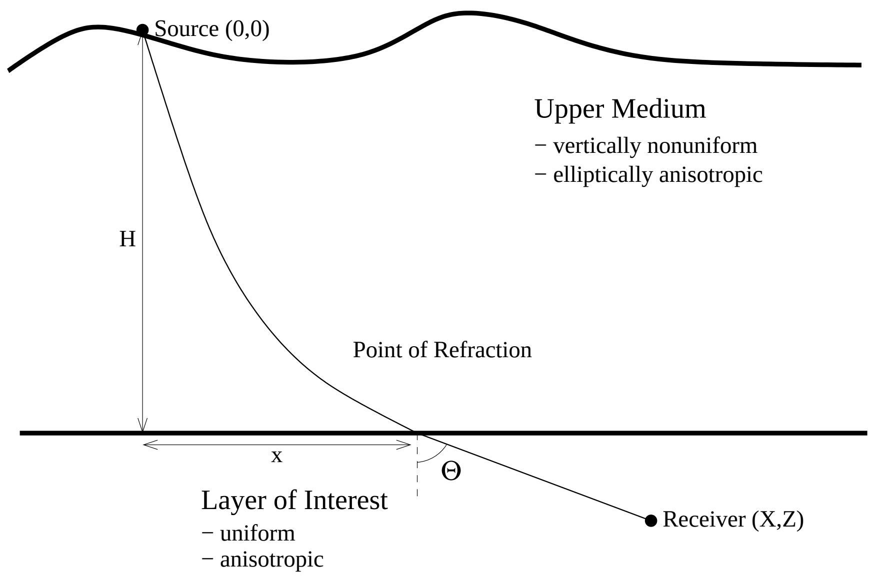


In this report, we consider the propagation of seismic waves through a medium that can be subdivided into of two distinct parts. The upper part is assumed to be azimuthally symmetric, linearly nonuniform with increasing depth, and the velocity dependance with direction consistent with elliptical anisotropy. The lower part, which is the layer of interest, is assumed to also be azimuthally symmetric, but uniform and nonelliptically anisotropic. Despite nonellipticity, we assume the angular dependance of the velocity can be described by a convex curve.

Our goal is to produce a single source-single receiver model which uses modern seismic measurements to determine the elastic moduli of the lower media. Once known, geoscientists could better describe the angular dependance of the velocity in the layer of interest and also would have some clues at to the actual material composing it.

4.2 Formulation of the Problem

We begin by assuming that the area of study consists of layered sedimentary rocks. Rotational symmetry in such a case effectively reduces problem to two dimensions. Since, in the case of rational symmetry, energy will propagate between source and receiver in a vertical plane, we can define our coordinate system such that our x and z coordinate axes contain the plane of propagation.

We start with a vertical cross section of the earth with a seismic source on the surface and a receiver located below. We define the horizontal offset between source and receiver to be X, and the vertical offset to be Z. That is, we define the source to be at the origin of our coordinate system, $(0, 0)$, and the receiver to be at the location (X, Z). Furthermore, we assume that the subsurface is divided into two distinct media with the depth to the interface between them to be H. Finally, we define the horizontal offset between the source and the point of refraction of the seismic ray to be x.

4.2.1 Assumptions

We assume that the seismic source is of high frequency with a wavelength shorter than all characteristic lengths in the layers of interest. With this assumption, it follows, from asymptotic ray theory, that the signal travels along a ray, which is a trajectory of the stationary traveltime. Thus, we assume that the rays obey the variational principle of Fermat.

We also assume that the medium through which the seismic waves propagate is a perfectly linear elastic continuum. Hence the stress-strain relationships are described by Hooke's law.

4.2.2 Upper Medium

We consider the upper medium, $0 \leq z \leq H$, to be horizontally stratified $\{(i.e.$, sedimentary rocks)$\}$ and both vertically nonuniform and elliptically anisotropic. Vertical nonuniformity is often the result of varying compaction of the subsurface and manifests itself as a linear increase in $\{$vertical$\}$ velocity with depth. That is, $V_z(z) = a + bz$. In the case of elliptical anisotropy we can define a single parameter, χ, that describes the relationship between vertical and horizontal velocity[1]. Specifically,

$$\chi = \frac{V_x^2 - V_z^2}{2V_z^2}. \tag{4.1}$$

Combining the effects of both nonuniformity and anisotropy results in a ray velocity profile[2] of the form

$$V(z, \Theta; a, b, \chi) = (a + bz)\sqrt{\frac{1 + 2\chi}{1 + 2\chi\cos^2(\Theta)}} \tag{4.2}$$

where a, b, and χ are predetermined.

Then, by applying Noether's Theorem it can be shown that the ray form of Snell's Law leads to the expression for the ray parameter

$$p = \frac{2x}{\sqrt{(x^2 + (1+2\chi)\,z^2)\,(4a^2\,(1+2\chi) + 4ab\,(1+2\chi)\,z + b^2\,(x^2 + (1+2\chi)\,z^2))}}. \tag{4.3}$$

Applying the variational principle of Fermat, given the above assumptions, the traveltime function[3] of a ray traveling from a source at $(0,0)$ to a point (x, H) is

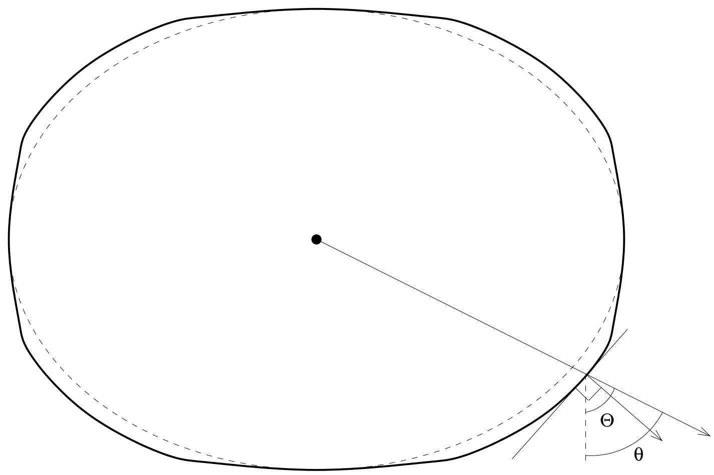
$$t_1(x, z; a, b, \chi) = \frac{1}{b}\ln\left|\frac{1 + \sqrt{1 - p(x)^2 a^2 (1+2\chi)}}{1 + \sqrt{1 - p(x)^2 (a+bz)(1+2\chi)}}\right|. \tag{4.4}$$

It has been assumed that the parameters a, b and χ, which characterize the upper medium, are known. These values are obtained by considering traveltimes from source to receivers located in the upper medium, and applying a best-fit method (*e.g.*, conjugate-gradient optimization) to the traveltime expression, t_1.

[1]Thomsen, L.,(1986) Weak Elastic Anisotropy, *Geophysics*, **51**, pp. 1954 - 1966.

[2]Affarwala, B., *et. al.*, (1997) Inversion of Anisotropic Velocity Parameter. *Proceedings of the first PIMS Industrial Solving Workshop*, PIMS, pp. 76-89.

[3]Slawinski, M., Wheaton, C., personal communication. 2002



4.2.3 Layer of Interest

We consider the layer of interest, $H \leq z \leq Z$, to be uniform, perfectly elastic, and transversely isotropic. Uniformity implies that all rays will be straight lines from the refraction point to the receiver. Elasticity and transverse isotropy imply that under Hooke's Law, which give the relationship between stress, σ, and strain, ϵ, via the elastic moduli, C_{ij},

$$\sigma_i = C_{ij}\epsilon_j, \quad i, j \in [1, ..., 6],$$

there will be five independent elasticity constants. Furthermore, if we focus solely on compressional-wave data, this number further reduces to four. In other words, only $C_{11}, C_{13}, C_{33}, C_{44}$ will be independent and need to be solved for in this particular case.

In the equations above, we have made reference to the ray angle, Θ. However, the angle that is measured in field observations is not the ray angle. In fact, in the presence of anisotropy there are three distinct angles associated with wave propagation which must be carefully distinguished. The first is the ray angle, Θ, which is associated with the direction of energy propagation. In the case of uniform media where raypaths are straight, it is equal to the angle between the vertical and the radian to a point on the wavefront (see Figure 2). If we were to {then} construct a perpendicular to the wavefront at end of the radius, the angle between the perpendicular and the vertical would be the phase angle, θ. Thirdly, we have the displacement (or polarization) angle, ϕ, which describes the direction of particle displacement as the wavefront propagates. This last angle is obtained by measuring three orthogonal displacements at the receiver. It can be shown that both the ray and polarization angles, Θ and ϕ, respectively, are related to the phase angle by the following expressions[4],

$$\tan(\Theta) = \frac{\tan(\theta) + \frac{1}{v}\frac{\partial v}{\partial \theta}}{1 - \frac{\tan(\theta)}{v}\frac{\partial v}{\partial \theta}} \tag{4.5}$$

[4]Arnold, V.I., (1989) Mathematical Methods of Classical Mechanics (2nd ed.): Springer-Verlag, pp. 248 - 252.

$$\tan(\phi) = \frac{\rho v^2 - C_{44}\sin^2(\theta) - C_{33}\cos^2(\theta)}{(C_{13}+C_{44})\sin(\theta)\cos(\theta)} \tag{4.6}$$

Our goal is to formulate and solve an inverse problem, in which a, b, χ, X, Z, T,and ϕ are the inputs and the elasticity constants, $C_{11}, C_{13}, C_{33}, C_{44}$, of the layer of interest are the outputs.

4.3 Method of Solution

4.3.1 Solving the Forward Problem

Before we attempt to tackle the inverse problem, we consider the forward problem. By starting with the forward problem we will be able to generate input values for the inverse problem while at the same time knowing what outputs the inverse should give. We begin by fixing the values of a, b, and χ for the upper medium. Also, we set the refraction point, x, the mass density of the lower layer of interest, ρ, the elastic moduli, C_{ij}, and the total traveltime, T,from source to receiver. Using the equations above, we are then able find the refraction point, x, and the polarization angle, ϕ.

First, we calculate the horizontal ray slowness, p, in the upper medium by substituting the refraction point (x, H) into equation (4.3)

$$p(x) = \frac{2x}{\sqrt{(x^2+(1+2\chi)H^2)(4a^2(1+2\chi)+4ab(1+2\chi)H+b^2(x^2+1+(1+2\chi)H^2}}$$

and also the traveltime in the upper medium using equation (4.4)

$$t_1(x, z; a, b, \chi) = \frac{1}{b}\ln\left|\frac{1+\sqrt{1-p(x)^2a^2(1+2\chi)}}{1+\sqrt{1-p(x)^2(a+bz)(1+2\chi)}}\right|.$$

By Snell's law, the tangential component of phase slowness, p, will be conserved across the interface. If θ and $v(\theta)$ are the phase angle and phase velocity, respectfully, for the lower layer we have

$$\frac{\sin(\theta)}{v(\theta)} = p.$$

The phase velocity[5], $v(\theta)$, is given by

$$v^2(\theta) = \frac{(C_{33}-C_{11})\cos^2(\theta)+C_{11}+C_{44}+\sqrt{\Delta}}{2\rho} \tag{4.7}$$

with

$$\begin{aligned}\Delta = {}& [(C_{11}-C_{33})\cos^2(\theta)-(C_{11}+C_{44})]^2 \\ & -4[C_{33}C_{44}\cos^4(\theta)-(2C_{13}C_{44}-C_{11}C_{33}+C_{13}^2)\cos^2(\theta)\sin^2(\theta)+C_{11}C_{44}\sin^4(\theta)].\end{aligned}$$

[5]Daley, P.F. & Hron, F., 1977. Reflection and Transmission Coefficients for Transversely Isotropic Media, *Bull. Seism. Soc. Am.*, **67**, 661-675.

At this point we can solve for the phase angle and thereby the phase velocity in the lower medium. Once we have the phase velocity, we can solve for the ray angle using equation (4.5),

$$\tan(\Theta) = \frac{\tan(\theta) + \frac{1}{v}\frac{\partial v}{\partial \theta}}{1 - \frac{\tan(\theta)}{v}\frac{\partial v}{\partial \theta}}.$$

We then substitute the phase angle into the ray velocity expression, can find the ray velocity, V, using

$$V^2(\theta) = v^2(\theta) + \left(\frac{\partial v}{\partial \theta}\right)^2. \tag{4.8}$$

Since the layer of interest is uniform, the ray path is a straight line and the ray velocity in the direction of the ray angle will be constant. Thus the traveltime, t_2, in the layer of interest is

$$t_2 = \frac{\sqrt{(X-x)^2 + (Z-H)^2}}{V(\Theta)} \tag{4.9}$$

and the total travel time, T , is

$$T = t_1(x) + t_2(x).$$

Substituting the phase angle into expression (4.6) we get the polarization angle, ϕ,

$$\tan(\phi) = \frac{\rho v^2 - C_{44}\sin^2(\theta) - C_{33}\cos^2(\theta)}{(C_{13} + C_{44})\sin(\theta)\cos(\theta)}.$$

4.3.2 Solving the Inverse Problem

Next we attempt to solve the inverse problem directly by combining the expressions described above into a system of equations. Since the rays will be straight in the lower medium we can reexpress the equations above to get the following system of nonlinear equations,

$$\begin{aligned} \frac{X-x}{Z-H} - \frac{\tan(\theta) + \frac{1}{v}\frac{\partial v}{\partial \theta}}{1 - \frac{\tan(\theta)}{v}\frac{\partial v}{\partial \theta}} &= 0 \\ T - \frac{\sqrt{(X-x)^2 + (Z-H)^2}}{V(\theta)} - t_1(x) &= 0 \\ p(x)v(\theta; C_{ij}) - \sin(\theta) &= 0 \\ \tan(\phi) - \frac{\rho v^2 - C_{44}\sin^2(\theta) - C_{33}\cos^2(\theta)}{(C_{13} + C_{44})\sin(\theta)\cos(\theta)} &= 0. \end{aligned} \tag{4.10}$$

Unfortunately we have the problem of an underdetermined system with six unknowns $(x, \theta, C_{11}, C_{13}, C_{33}, C_{44})$ but only four equations. However, the industrial contacts on the team tell us that reasonable estimates of C_{33} and C_{44} are not completely out of reach. Therefore we proceed with the added the restriction that C_{33} and C_{44} are both known.

This system can now be solved using either Jacobi Iteration or the Conjugate-gradient Method. Results

As a starting point for the forward problem, we used the following the parameter values which are typical in actual settings,

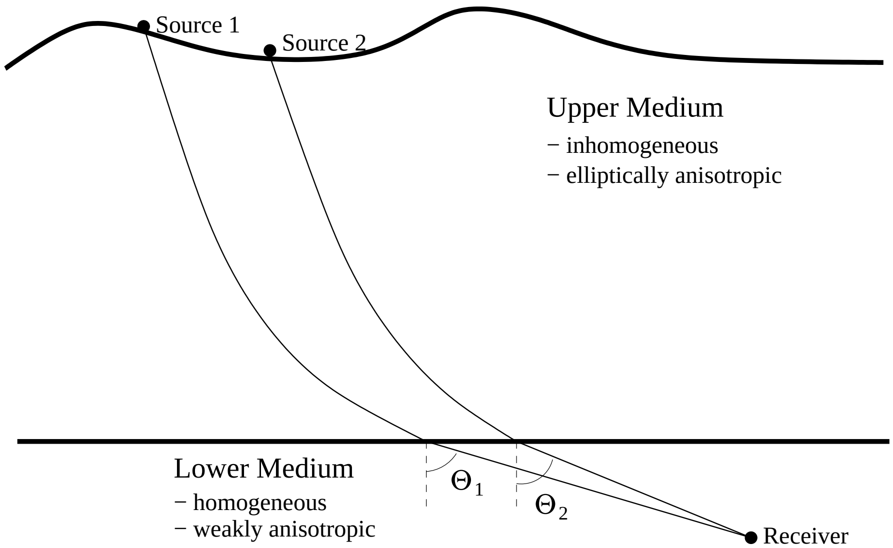
$$\begin{array}{ccc} a = 2000 & C_{11} = 3.13e10 & Z = 1154.46 \\ b = 0.3 & C_{13} = 0.34e10 & X = 1057.03 \\ \chi = 0.2 & C_{33} = 2.25e10 & H = 700. \\ \rho = 2310 & C_{44} = 0.65e10 & \end{array}$$

Using the forward problem, we generated the corresponding values of T, ϕ. Then, we used our inverse problem code to try to regenerate the values for $x, C_{11}, C_{13}, C_{33}, C_{44}$. For exact values of the input values, T, ϕ, the system converged to the correct values of C_{ij}. Unfortunately however, for even small perturbations of the inputs (consistent with experimental error) a solution was often not possible and when solutions were found, they tended to be widely different from the true values. Due to the time constraints of the workshop we were not able proceed with an in-depth error analysis. However, it was noted that provided the experimental errors were random (as opposed to systematic) a collection of outputs obtained by considering numerous source-receiver setups, could be used to estimate the true C_{ij}'s. By taking the best fit for each set of estimates of C_{ij}, the results tended to be reasonably close to the true values.

4.4 Future Work

Through our work on this problem we feel that our progress was marked by "raising more questions than answers". As a starting point for future or continued work on this project we would suggest that an in-depth error analysis be performed. By doing so, hidden redundancies in the formulations and suitable initial/boundary conditions may be identified to combat the sensitivity to errors and non uniqueness of solutions.

More importantly, however, we feel that the problem would be best approached by considering not one, but two sources on the surface (Figure 3). By doing so this would result in eight equations in eight unknowns, negating the need for *a priori* knowledge of C_{33} and C_{44}.



Adding a second source to equations (4.10) would give,

$$
\begin{aligned}
0 &= \frac{X - x_1}{Z - H} - \frac{\tan(\theta_1) + \frac{1}{v_1}\frac{\partial v_1}{\partial \theta_1}}{1 - \frac{\tan(\theta_1)}{v_1}\frac{\partial v_1}{\partial \theta_1}} \\
0 &= \frac{X - x_2}{Z - H} - \frac{\tan(\theta_2) + \frac{1}{v_2}\frac{\partial v_2}{\partial \theta_2}}{1 - \frac{\tan(\theta_2)}{v_2}\frac{\partial v_2}{\partial \theta_2}} \\
0 &= T_1 - \frac{\sqrt{(X - x_1)^2 + (Z - H)^2}}{V_1(\theta_1)} - t_1(x_1) \\
0 &= T_2 - \frac{\sqrt{(X - x_2)^2 + (Z - H)^2}}{V_2(\theta_2)} - t_2(x_2) \\
0 &= p_1(x_1)v_1(\theta_1; C_{ij}) - \sin(\theta_1) \\
0 &= p_2(x_2)v_2(\theta; C_{ij}) - \sin(\theta_2) \\
0 &= \tan(\phi_1) - \frac{\rho v_1^2 - C_{44}\sin^2(\theta_1) - C_{33}\cos^2(\theta_1)}{(C_{13} + C_{44})\sin(\theta_1)\cos(\theta_1)} \\
0 &= \tan(\phi_2) - \frac{\rho v_2^2 - C_{44}\sin^2(\theta_2) - C_{33}\cos^2(\theta_2)}{(C_{13} + C_{44})\sin(\theta_2)\cos(\theta_2)}
\end{aligned}
\tag{4.11}
$$

4.5 Sample File Used to Run Simulations

Throughout the coding of the equations presented above, a combination of Matlab^{TM} and Maple^{TM} code was used (depending on who owned the laptop we were using at the time). Following is the coding done using Maple ^{TM} to manipulate the input values as stated above. The first code was used to input values to be solved for in the inverse model, and generate data for the inverse model.

```
>restart:Digits:=40:
```

Forward Model where Cij's Known and Refraction Point (xr,zr) Known.

```
>
># In this program we leave the Z of the receiver position a free parameter and solve for it in the process. We only set the horizontal offset of the receiver with respect to the refraction point, which we fix. The origin is assumed to be at the source.
># Enter in the input values and solve for receiver location, total traveltime, and polarization angle at receiver
>a:=2000: b:=0.8: chi:=0.3: zr:=800: roe:=2310: # zr=depth of interface/refraction point.
>C11:=3.13*10^10: C13:=0.34*10^10: C33:=2.25*10^10: C44:=0.65*10^10:
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&
>X:=2600: xr:=1154.46:          # X=receiver depth, xr= refraction point.
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&
>
># Traveltime in the upper medium
>p:=2*xr/sqrt((xr^2+(1+2*chi)*zr^2)*(a^2*(4+8*chi)+4*a*b*(1+2*chi)*zr+b^2*(xr^2+(1+2*chi)*zr^2))):
>t1:=(1/b)*ln( (a+b*zr)/a * (1+sqrt(1-p^2*a^2*(1+2*chi)))/(1+sqrt(1-p^2*(a+b*zr)^2*(1+2*chi)))):
>
># Calculate the phase velocity and its derivative with respect to phase angle, theta
>Delta:=((C11-C33)*cos(theta)^2-(C11+C44))^2-4*(C33*C44*cos(theta)^4-(2*C13*C44-C11*C33+C13^2)*
cos(theta)^2*sin(theta)^2+C11*C44*sin(theta)^4):
>v:= sqrt((C11+C44+(C33-C11)*cos(theta)^2+sqrt(Delta))/(2*roe)):
```

The Phase Angle for the ray traveling down from the refraction point (xr,zr):

will have problems if p*v >1. arcsin will then be complex (ie, Shadow Zone.) When you fix the refraction point, snell's law must have a solution for theta (a phase angle must exist). Since you are using p, conserved from upper medium, combined with v(theta) for the lower medium, if your refraction point is such that your angle of incidence is beyond the critical angle, snell's law will not have a solution in the lower medium because there will be NO TRANSMITION.

```
>eqn:=theta=arcsin(p*v):
>theta:=fsolve(eqn,theta,0..Pi/2);
```

```
>
># Solve for the Ray Angle from refraction point to receiver at (X,Z).
>rayang:=arctan((tan(theta)+1/v*dv)/(1-tan(theta)/v* dv)):
>rayangle:=evalf(%):
>
```

```
># Calculate the polarization angle along the ray in the lower medium
>phi:=arctan( (roe*v^2-C44*sin(theta)^2-C33*cos(theta)^2)/((C13+C44)*sin(theta)*cos(theta))):
>
># Solve for the Ray velocity
>rayvelocity:=sqrt(v^2+dv^2):
```

Solve for the traveltime in the lower medium, t2.

Start by fixing the offset of x-coordinate of the receiver position from the refraction point, xr, than calculate the traveltime and z-position of receiver.

```
>xo:=X-xr:
```

```
>w:=xo/sin(rayangle):
>t2:=w/rayvelocity:
>Z:=zr+xo*cot(rayangle):
>T:=t1+t2:
```

==

Data to be used for the inverse problem is as follows,

C33, C44 Values;

```
>C33:=C33; C44:=C44;
```

Receiver location,

```
>X,Z:=X,Z;
```

Total Traveltime...

```
>TravelTime:=T;
```

And Polarizaton at the receiver is....

```
>phi:=phi; evalf(phi*180/Pi);
```

==

Inverse Code Should Give,

```
>xr:=xr; C11:=C11; C13:=C13;
```

The second code that is presented is a formulation of the inverse problem which relies on Maple^{TM}'s "fsolve" numerical solver to determine the values of the elastic moduli.

Inverse Model for Traveltimes and Polarization Angles for 1 Source / 1 Receiver Setup.

```
>

># List the Input Data for the inverse problem.

>restart:

>Digits:=40:

>a:=2000: b:=0.8: chi:=0.3: zr:=700: roe:=2310:

>C33:=2.25*10^10: C44:=0.65*10^10:

>X := 1057.0356:

>Z := 729.57398:

>T1 := .48321603:

>phi1 := 1.24456:

>p1:=2*xr1/sqrt((xr1^2+(1+2*chi)*zr^2)*(a^2*(4+8*chi)+4*a*b*(1+2*chi)*zr+b^2*(xr1^2+(1+2*chi)*zr^2))):

>Delta1:=((C11-C33)*cos(theta1)^2-(C11+C44))^2-4*(C33*C44*cos(theta1)^4-(2*C13*C44-C11*C33+C13^2)*
cos(theta1)^2*sin(theta1)^2+C11*C44*sin(theta1)^4):

>v1:=sqrt((C11+C44+(C33-C11)*cos(theta1)^2+sqrt(Delta1))/(2*roe)):

>dv1:=diff(v1,theta1):

>t1a:=(1/b)*ln(abs((a+b*zr)/a*(1+sqrt(1-p1^2*a^2*(1+2*chi)))/(1+sqrt(1-p1^2*(a+b*zr)^2*(1+2*chi))))):

># System of Equations to solve for C11,C13,C33,C44,xr1,theta1

>eqn1:=v1*p1-sin(theta1)=0:

>eqn2:=T1-sqrt(((X-xr1)^2+(Z-zr)^2)/(v1^2+dv1^2))-t1a=0:

>eqn3:= tan(phi1)=( (roe*v1^2-C44*sin(theta1)^2-C33*cos(theta1)^2)/((C13+C44)*sin(theta1)*cos(theta1))):

>eqn4:= (X-xr1)/(Z-zr)=((tan(theta1)+1/v1*dv1)/(1-tan(theta1)/v1*dv1)):

>sys:=[eqn1,eqn2,eqn3,eqn4]:

>sols:=fsolve({eqn1,eqn2,eqn3,eqn4},{xr1,C11,C13,C33,C44,theta1});

>assign(sols);

>xr1:=xr1; C11:=C11; C13:=C13; C33:=C33; C44:=C44;
```

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[1] Affarwala, B., *et. al.*, (1997) Inversion of Anisotropic Velocity Parameter. *Proceedings of the first PIMS Industrial Solving Workshop*, PIMS, pp. 76-89.

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[5] Slawinski, M., (2002) Waves and Rays in Elastic Continua, in print.

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