Symmetry Breaking in Jetting

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Abstract

In the bubble-jet printing process, it has been observed that the drop that ultimately pinches off from the ink jet sometimes moves sideways rather than straight relative to the symmetry axis of the liquid jet. Our group examined various mechanisms that might lead to the deflection of the ink drop. In particular, we focused on whether the liquid filament that connects the lead drop to the nozzle is capable of supporting lateral waves which might propagate from the nozzle toward the lead drop and break the symmetry at pinch-off.

1 Introduction

We were introduced to the design of an ink-jet printer manufactured by Xerox [1]. The print-head consists of an array of parallel cylinders, each containing ink. The cylinders have square cross-section (approximately 30 μm in size), one end being connected to the ink supply tank and the other end connected to a nozzle which tapers down to a small circular hole (nozzles of square, triangular, elliptical and other cross-sections are also in use). In the wall of the cylinder near the taper there is an electric heating element. A pulse of current through this element causes an almost instantaneous vaporization of the ink in its proximity. This gaseous vapor bubble forces the ink between it and the hole at the cylinder end to be ejected through the hole. The ink vaporizes and the ink is ejected within a few microseconds. The refilling process takes tens of microseconds. The maximum pressure reached within the vapor bubble is about 50 atmospheres, occurring within a fraction of a microsecond. The velocity of the ejected drop is anywhere in the range 1–40 m/s. The volume of the main drop is about 10 picoliters [1].

Quality printing will be achieved if: (a) The ink which issues from the hole as a jet forms a drop at a forward location, which then severs from the filament connecting it to the cylinder, (b) the drop is propagated along the intended direction, i.e., the symmetry axis of the cylinder, and (c) the ink left behind retracts back into the cylinder under the suction and surface tension forces caused by the rapid collapse of the ink vapor bubble.

However the scenario set out in (a), (b) and (c) is not always obtained. Other scenarios in evidence include: (i) The filament trailing from the lead drop adopts a wavy form, breaks up into a number of smaller, satellite drops which scatter off the symmetry axis. (ii) The lead drop propagates at an angle to the symmetry axis.

Xerox introduced the problem of identifying the mechanism(s) causing (ii). The misdirection of drops clearly is a cause of low-quality printing. Unfortunately there was little reproducibility of the circumstances causing (ii); one cylinder would misdirect for a few drops, then would re-establish the design behavior. No change in parameter values could consistently cause misdirection.

A variety of mechanisms for symmetry breaking were suggested. These were divided into two broad categories: those somehow associated with surface tension, and those not. The latter category included possible action of an aerodynamic "lift" force on the drop that would act if the drop were spinning after pinch-off, an asymmetric blockage of the nozzle leading to the jet being deflected at the nozzle itself, or the non-uniformity of the velocity profile in the fluid that reaches the tapered nozzle. Possible causes associated with surface tension included the existence of lateral waves (supported by the tension) in the filament that connects the drop to the nozzle initially, off-center pinch-off of the lead drop from the filament, asymmetry in the retraction of the ink into the nozzle during the collapse of the vapor bubble, and meniscus asymmetry prior to ejection due to wetting non-uniformities at the rim of the nozzle. In our group effort, we focused on the question of lateral waves on the filament and on whether asymmetry in the retraction of the ink into the nozzle could create waves that propagate along the filament and affect the lead drop. The mathematical model we proposed to analyze these effects is introduced below and the results of our analysis are given in separate sections. Some of the existing literature on breakup of liquid jets [2, 3, 4, 5, 6] is listed in the bibliography at the end of this report.

2 Mathematical Model

Consider an inviscid liquid jet moving primarily in the horizontal direction (denoted by x), whose cross-sectional area is assumed to remain circular, with radius $h(x,t)$, but whose centerline is allowed to have small vertical deflections from the x-axis. Denote the position of the centerline of the jet by $y(x,t)$ and let the density of the liquid jet and its surface tension be given by ρ and σ, respectively. The average horizontal velocity of the jet at position x as a function of time t is denoted by $u(x,t)$. For an incompressible liquid, the differential form of the law of conservation of volume of the jet thus takes the form

$$\frac{\partial h^2}{\partial t} + \frac{\partial (uh^2)}{\partial x} = 0\,. \tag{1}$$

The pressure, $p(x,t)$, within the slender jet is related to the ambient pressure, p_o, by the Young-Laplace equation

$$p = p_o + \sigma \left[\frac{1}{h} - \frac{\partial^2 h}{\partial x^2} \right] , \tag{2}$$

where the two terms within the brackets represent the principal radii of curvature of the jet surface, one within the circular cross-section of radius h, and the other in the vertical plane passing through the jet. The pressure is assumed to be uniform across the jet cross-section. As such, the horizontal momentum equation for the jet is given by

$$\frac{Du}{Dt} = \frac{-1}{\rho}\frac{\partial p}{\partial x} = \frac{\sigma}{\rho}\frac{\partial}{\partial x}\left[\frac{\partial^2 h}{\partial x^2} - \frac{1}{h} \right] . \tag{3}$$

Here,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{4}$$

is the substantial time-derivative.

The small vertical deflections of the jet from the symmetry axis are governed by a vertical momentum equation that is quite similar to the equation for a taut string under tension. In the case of the jet, the equivalent tension is given by $2\pi h\sigma$ (the circumference of the jet multiplied by the surface tension) and the vertical acceleration of the jet is given by Dy/Dt. Applying Newton's second law to a segment of the jet between x and $x + \Delta x$ yields:

$$\frac{D}{Dt}\left[\rho \pi h^2 \Delta x \frac{Dy}{Dt} \right] = \left[2\pi h \sigma \frac{\partial y}{\partial x} \right]\Big|_{x+\Delta x} - \left[2\pi h \sigma \frac{\partial y}{\partial x} \right]\Big|_{x} . \tag{5}$$

Dividing by Δx and taking the limit $\Delta x \to 0$ then yields

$$\frac{D}{Dt}\left[h^2 \frac{Dy}{Dt} \right] = \frac{2\sigma}{\rho}\frac{\partial}{\partial x}\left[h \frac{\partial y}{\partial x} \right] . \tag{6}$$

Eqs. (1), (3) and (6) constitute the complete set of partial differential equations that describe the dynamics of the jet of ink, under the assumptions that the jet remains circular, its radius varies "slowly" with respect to x (i.e., on a length scale much larger than its radius) and its centerline has small vertical deflections from the horizontal axis. Starting with this set of equations, we now examine several problems that provide insight into the behavior of the jet and sideways pinch-off of the lead drop.

3 Lateral Waves on a Uniformly Stretching Filament

Consider the situation where the relatively massive lead drop has been given an impulsive motion with constant velocity U_o. A thin cylindrical filament of initial length x_o trails behind and connects the drop to the nozzle (at $x = 0$), see Fig. 1. Without any perturbations, the cylindrical filament would stretch to length

$$X(t) = x_o + U_o t \,, \tag{7}$$

at time t, and the velocity along the filament would vary linearly with distance according as

$$u(x,t) = \frac{U_o\, x}{X(t)} \,. \tag{8}$$

Conservation of volume then dictates that

$$h^2(t)X(t) = h_o^2 x_o \,, \tag{9}$$

where h_o is the initial radius of the cylindrical filament. This uniformly stretching and thinning motion of the filament represents an exact solution to the conservation of volume and x-momentum equations, (1) and (3).

Starting with this "base flow," we now allow for small deflections of the centerline of the filament,[1] $y(x,t)$, and obtain the following equation for the evolution of y:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left[h^2\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)y\right] = \frac{2\sigma}{\rho}\frac{\partial}{\partial x}\left(h\frac{\partial y}{\partial x}\right) \,, \tag{10}$$

in which $u(x,t)$ and $h(t)$ are given by Eqs. (8) and (9).

Starting with this equation we now ask whether a lateral disturbance of the centerline initiated at the nozzle ($x = 0$) can possibly reach the lead drop that is at position $X(t)$. Thinking of the filament as a taut string with tension $2\pi h\sigma$ and with mass per unit length $\pi h^2\rho$, the wave-speed along the filament would be approximated by $(2\sigma/h\rho)^{1/2}$ which increases as the filament radius decreases. Since the velocity of the lead drop remains constant, any wave generated at the nozzle should thus ultimately reach the drop, unless the drop pinches off earlier.

[1] A lateral disturbance could be caused by a non-symmetric retraction of the filament as the ink is being sucked into the nozzle.

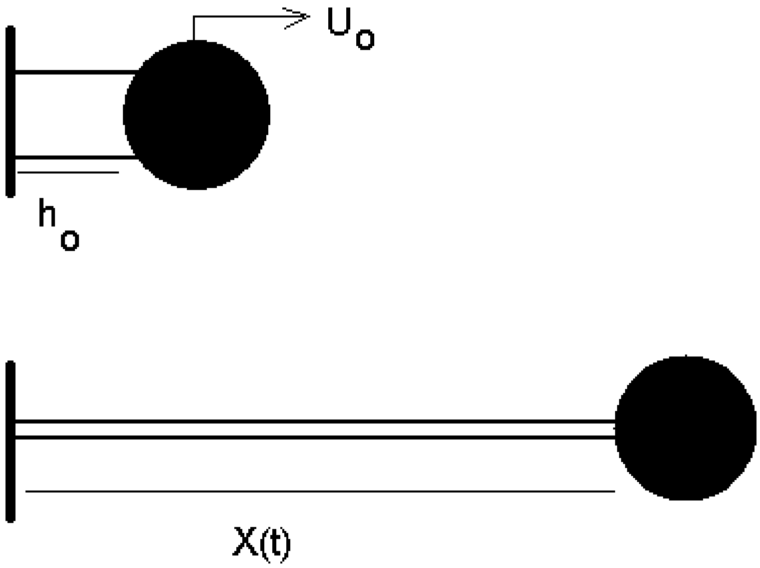

Figure 1: Sketch of the uniformly stretching filament.

To analyze this issue further, we effect the change of variables from (x, t) to (ξ, s) where

$$
\begin{aligned}
s &= t && (11)\\
\xi &= x x_o/(x_o + U_o t) && (12)
\end{aligned}
$$

thereby transforming Eq. (10) to the form

$$\frac{1}{X}\frac{\partial}{\partial s}\left(\frac{1}{X}\frac{\partial y}{\partial s}\right) = c_o^2 \frac{x_o^{3/2}}{X^{7/2}}\frac{\partial^2 y}{\partial \xi^2}\,. \tag{13}$$

In this equation

$$c_o^2 = \frac{2\sigma}{\rho h_o}\,, \tag{14}$$

which is the square of the wave-speed corresponding to the initial radius of the filament. The additional change of variable

$$\tau = x_o s + \frac{U_o s^2}{2} \qquad \text{so that} \qquad \frac{\partial}{\partial \tau} = \frac{1}{X}\frac{\partial}{\partial s} \tag{15}$$

simplifies the last wave equation to the form

$$\frac{\partial^2 y}{\partial \tau^2} = c_o^2 \frac{x_o^{3/2}}{X^{7/2}}\frac{\partial^2 y}{\partial \xi^2}\,, \tag{16}$$

which is a standard wave equation with a time-dependent wave-speed.

The equation for the right-going characteristics is given by

$$\frac{d\xi}{d\tau} = c_o \frac{x_o^{3/4}}{X^{7/4}}\,, \tag{17}$$

with $\xi = 0$ at $\tau = 0$ the proper initial condition to find that characteristic which starts at the nozzle. In terms of t, this is given by

$$\frac{d\xi}{dt} = c_o \frac{x_o^{3/4}}{X^{3/4}} \qquad \text{with} \qquad x(0) = 0\,. \tag{18}$$

This is solved by

$$\frac{\xi(t)}{x_o} = \frac{4c_o}{U_o}\left[\left(1 + \frac{U_o t}{x_o}\right)^{1/4} - 1\right]. \tag{19}$$

This characteristic reaches the lead drop when the left-hand side of the last equation equals unity. This occurs at time

$$t^* = \frac{x_o}{U_o}\left[\left(\frac{U_o}{4c_o} + 1\right)^4 - 1\right]. \tag{20}$$

Apparently, the lead characteristic starting at the nozzle *always* reaches the lead drop, since the filament is getting thinner over time and the wave-speed of lateral waves on the filament continually increases, whereas the velocity of the lead drop remains constant. However, if the drop pinches off before any such disturbance reaches it, there will not be any symmetry breaking due to such lateral disturbances originating from the nozzle. If the characteristic time for pinch-off is denoted by t_{po}, the main conclusion is that the ratio t^*/t_{po} determines whether the lead drop would travel in a straight path or if it might be deflected due to lateral waves on the filament breaking the symmetry prior to pinch-off. (See the last part of the Conclusions section for the orders of magnitude of some of these parameters.)

4 A Travelling Wave Description of the Receding Tail

We now consider whether the continuity and x-momentum equations, (1) and (3), which we derived earlier can be used to actually describe the profile of the tail of the filament which is receding into the nozzle. For this purpose, suppose that the centerline of the jet is straight ($y = 0$) and that the nozzle is in the form of a long cylindrical tube of radius h_o. Imagine the jet to be receding in a steady fashion into the nozzle, with the jet interface making a specified contact angle with the nozzle wall, see Fig. 2. Denote by V the velocity at which the contact angles are receding along the tube. Upon scaling h and x with h_o, and u with V, in a frame of reference which is itself receding with speed V so that the jet appears to be stationary, the dimensionless jet profile $h(x)$ and horizontal velocity $u(x)$ satisfy the pair of ODEs

$$uu' = \frac{1}{\mathcal{W}}(h'' - h^{-1})'\,, \tag{21}$$

$$(uh^2)' = 0\,, \tag{22}$$

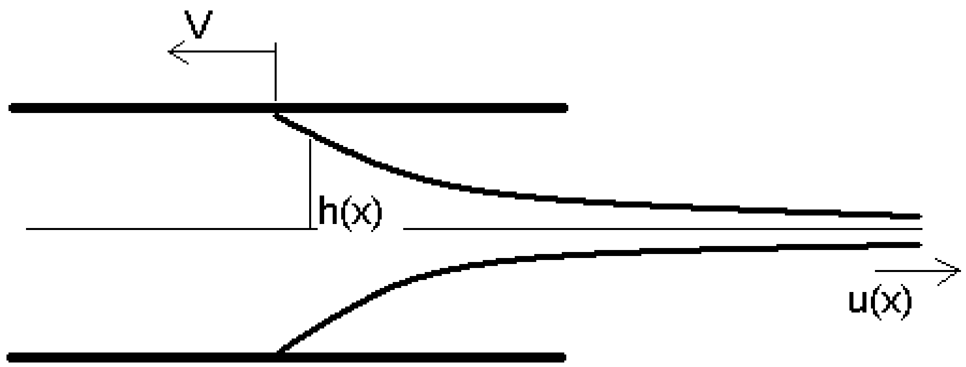

Figure 2: Sketch of the receding jet of ink.

derived from Eqs. (1) and (3). These are to be solved subject to the conditions that at $x = 0$ (which is the position where the contact lines touch the walls of the tube):

$$u(0) = 1\,, \qquad h(0) = 1\,, \qquad h'(0) = -\alpha\,. \tag{23}$$

In these equations, $\mathcal{W} = \rho V^2 h_o/\sigma$ is the Weber number based on the velocity at which the jet is receding, and α is related to the (receding) contact angle of the ink jet along the nozzle. Integration of (22) yields $uh^2 = 1$ using which u can be eliminated from the equations to yield a single nonlinear second order ODE for $h(x)$:

$$h''(x) = \frac{1}{h} + \frac{\mathcal{W}}{2h^4} + \mathcal{C}\,, \tag{24}$$

with

$$h(0) = 1 \quad \text{and} \quad h'(0) = -\alpha\,. \tag{25}$$

Here, $\mathcal{C}$ is a constant of integration. If we analyze this second-order ODE by considering its two-dimensional phase plane (h, h'), we find that for negative values of $\mathcal{C}$, the system possesses a fixed point which turns out to be a center, with the trajectories forming closed paths around that center. For positive values of $\mathcal{C}$, there is no fixed point and all trajectories are open in the phase plane. In either case, starting at $x = 0$, $h(x)$ appears to decrease from 1 to reach a minimum value and either oscillate beyond that point (for the case when there is a fixed point, see Fig. 3) or increase without bound (in the other case, see Fig. 4). That $h(x)$ reaches a minimum value turns out to be important in determining whether disturbances generated at the contact lines can affect the pinch off of the lead drop connected to the jet. We examine this issue in the next section.

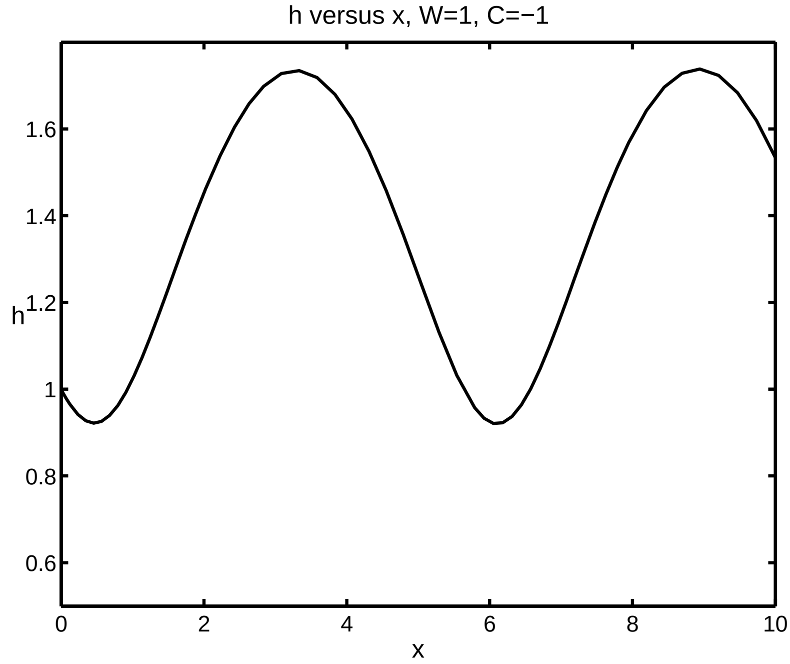

Figure 3: h vs. x at $\mathcal{W} = 1$, $\mathcal{C} = -1$, $\alpha = \pi/10$.

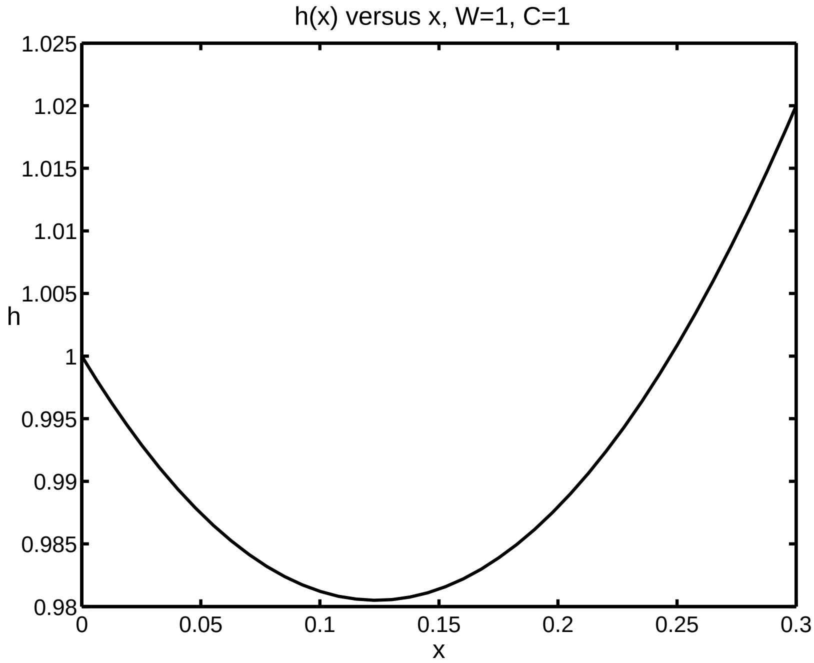

Figure 4: h vs. x at $\mathcal{W} = 1$, $\mathcal{C} = 1$, $\alpha = \pi/10$.

5 Lateral Waves on the Receding Tail

Given the position dependent radius and velocity profiles of the jet, $h(x)$ and $u(x)$, from the last section, we can examine the characteristics of lateral wave motion on the jet using our vertical momentum equation (6). Substituting the steady solutions $h(x)$ and $u(x)$ into Eq. (6), the result can be written as:

$$h^2 \left\{ \frac{\partial^2}{\partial t^2} + \frac{2}{h^2} \frac{\partial^2}{\partial t \partial x} + \left(\frac{1}{h^4} - \frac{2}{\mathcal{W} h} \right) \frac{\partial^2}{\partial x^2} \right\} y = \text{lower order derivatives}\,. \tag{26}$$

Since the operator on the left-hand side is of the generic form

$$\mathcal{L} = a \frac{\partial^2}{\partial t^2} + 2b \frac{\partial^2}{\partial t \partial x} + c \frac{\partial^2}{\partial x^2}\,, \tag{27}$$

its characteristics are the solutions to

$$\frac{dx}{dt} = -b \pm \sqrt{b^2 - ac}\,. \tag{28}$$

For our problem, this reduces to:

$$\frac{dx}{dt} = \frac{-1}{h^2} \pm \sqrt{\frac{2}{\mathcal{W} h}}\,. \tag{29}$$

The minus sign on the right-hand side describes waves that move to the left, whereas the plus sign might correspond to waves moving to the right, which are the ones we are interested in. Therefore, depending upon whether the combination

$$\frac{-1}{h^2} + \sqrt{\frac{2}{\mathcal{W} h}} \tag{30}$$

is smaller or greater than zero, some waves may be able to propagate forward, toward the lead drop. We thus find a critical value of h, given by

$$h^* = \left(\frac{\mathcal{W}}{2} \right)^{1/3}\,, \tag{31}$$

such that, for $h < h^*$ the characteristic speeds are negative while for $h > h^*$, they are positive. This has an interesting implication. Namely, for a given Weber number $\mathcal{W}$, if the jet narrows to a thickness less than the critical value h^* at some point after separating from the nozzle, no waves generated at the nozzle itself can pass through that point. The smaller the Weber number, the less likely it will be for the jet to become sufficiently thin to satisfy this condition. For a large Weber number, however, this criterion is more easily satisfied. Therefore, for stable symmetric operation of the ink jet printer, it is best to design for a large Weber number.

6 Conclusions

The key results of our analysis are as follows: For the case when the lead drop moves away from the nozzle at constant speed U_o, but remains connected to the nozzle through a thinning cylindrical ink filament, in order to avoid asymmetric pinch-off, the pinch-off time t_{po} must be designed to remain smaller than the value t^* given by

$$t^* = \frac{x_o}{U_o}\left[\left(\frac{U_o}{4c_o}+1\right)^4 - 1\right], \tag{32}$$

where x_o is the initial length of the cylindrical filament and $c_o = \sqrt{2\sigma/\rho h_o}$ is the wave-speed on the filament at the initial thickness h_o. To make t^* large (compared to t_{po}), the ratio U_o/c_o should be made as large as possible. The square of ratio is

$$\left(\frac{U_o}{c_o}\right)^2 = \frac{\rho U_o^2 h_o}{2\sigma}, \tag{33}$$

which is simply a Weber number based on the initial velocity of the lead drop.

At the same time, if another Weber number,

$$\mathcal{W} = \frac{\rho V^2 h_o}{\sigma}, \tag{34}$$

defined based on the speed V at which the ink jet is receding into the nozzle and the nozzle radius h_o is made large enough, it may be possible to prevent any disturbance generated at the nozzle to reach the lead drop. Both of these criteria suggest that designers of ink-jet printers should aim for large Weber number operation. That is, the lead drop must be given a sufficiently large impulsive velocity, and the ink and nozzle should be designed so that the volume of ink that is not ejected in the form of a drop can recede quickly back into the reservoir behind the nozzle.

Finally, note that using the characteristic values: $\rho = 1000$ kg/m^3, $\sigma = 0.07$ N/m, $h_o = 1.5 \times 10^{-5}$ m and $U_o = 5$ m/s, the initial velocity of lateral waves on the filament would be approximately $c_o \approx 3$ m/s, leading to $c_o/U_o \approx 0.6$ and $We = \rho U_o^2 h_o/\sigma \approx 2.7$. The velocity of the lead drop itself is consistent with an estimate one could obtain by taking the "impulse" delivered by the initial pressure (e.g., a pressure of 10 atmospheres acting over a 1 μs time interval, on an area given by πh_o^2), divided by the mass of the slug of ink that is ejected, $\pi h_o^2 L\rho$, with $L = 100$ μm. This calculation yields a velocity of about 10 m/s which is consistent with observations. We also estimated the Reynolds number of the translating drop based on the viscosity of the surrounding air and found it to be about 10, suggesting that in the aerodynamics inertia is important but not dominant. Our model, however, did not include viscous effects.

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