Chapter 5

Inversion for Anisotropic Velocity Parameter

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1 Introduction

This report describes the mathematical results of a team of seven researchers working under the auspices of the 1997 PIMSIPS Workshop, organized in Vancouver by the Pacific Institute for the Mathematical Sciences. The problem under study was raised by PetroCanada, and concerns the robust computation of a certain parameter of anisotropy from observed traveltimes of a seismic shear wave propagating through a geological medium.

2 Problem Description

A routine simplifying assumption in geophysical studies is that the velocity of a seismic wave in a given layer of material is independent of the direction of propagation; such a material is said to be isotropic. While this assumption greatly simplifies the mathematics describing certain geophysical phenomena, it ignores the physical reality that many materials which occur in geological layers are

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not isotropic. Understanding and measuring this so-called velocity anisotropy plays a key role in the development of a more accurate geophysical model.

Researchers at PetroCanada have developed an algorithm for computing a measure of anisotropy from recordings of traveltime data of seismic signals traversing various paths through horizontally layered media. The algorithm is based on an approximation to an exact physical model, and assumes an elliptical velocity profile in the anisotropy. The main problem addressed in this project is that the resulting approximate formula for anisotropy is extremely sensitive to all input parameters, in particular to the traveltime measurements which form the standard set of geophysical data.

PetroCanada is seeking to understand why this method is so sensitive to input data. Some possibilities are that the numerical methods used are not robust, or that the approximation is a source of error. More generally, they wish to find alternative, more robust methods of computing the anisotropy parameter.

3 Background Material

PetroCanada provided us with two references as background material: the 1996 Ph.D. thesis entitled "On elastic-wave propagation in anisotropic media: ... " by Michael Slawinski; and an abstract of a presentation to the recent SEG International Exposition entitled "Analytic inversion for Thomsen's γ parameter in weakly anisotropic media" by Michael Slawinski and Raphael Slawinski.

Within these works, the mathematics describing ray propagation through anisotropic media is presented without derivation. A first step is a reduction of the problem to two dimensions, which is typical in certain seismic imaging situations where all measurements are carried out in one plane. In two dimensions, the direction of propagation may in principle be described by a single angle. However, there are in fact two angles, the phase and group angles θ_p and θ_g, which are of interest in the mathematical formulation, along with the related phase and group velocities c_p and c_g. The anisotropy of a medium is indicated by Thomsen's parameter γ, from which one may describe the elliptical profile of the velocity, as a function of the phase angle, by the formula

$$c_p(\theta_p) = \beta\sqrt{1 + 2\gamma \sin^2 \theta_p}. \tag{1}$$

Here, β is the vertical velocity of propagation. The group velocity, as a function of phase angle, is given by

$$c_g(\theta_p) = \beta\sqrt{\frac{1 + 2\gamma(1+\gamma)(1 - \cos 2\theta_p)}{1 + \gamma(1 - \cos 2\theta_p)}}, \tag{2}$$

and the phase and group angles are related by

$$\tan \theta_g = (1 + 2\gamma) \tan \theta_p. \tag{3}$$

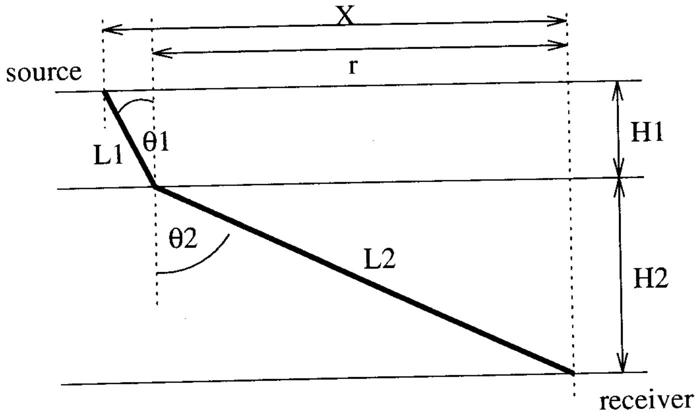

Figure 1: Two-layer model

Figure 1: Propagation of a seismic wave through a two layer medium.

An approximation is then made, assuming γ is small, so that

$$\theta_p \approx \theta_g \tag{4}$$

and

$$c_g \approx c_p \approx \beta(1 + \gamma \sin^2 \theta_p). \tag{5}$$

The inversion formula for γ is based on this approximation.

The physical model considered in the reference work is propagation of a seismic wave through a two layer medium as illustrated in Figure 1. The upper layer is isotropic, with velocity parameter c_1, while the lower layer is anisotropic with vertical velocity parameter β and anisotropic velocity parameter γ. A source on the surface, at some distance X from a vertical bore hole, generates a seismic signal which travels to a receiver at the bottom of the bore hole along a bent ray that refracts at the interface between the two layers some distance r from the bore hole. Noting that $r/L_2 = \sin\theta_2$ is the sine of the angle of propagation, one finds the traveltime for the signal is

$$\begin{aligned} t &= \frac{L_1}{c_1} + \frac{L_2}{c_2} \\ &\approx \frac{\sqrt{(X-r)^2 + H_1^2}}{c_1} + \frac{\sqrt{r^2 + H_2^2}}{\beta\,(1 + \gamma r^2/(r^2 + H_2^2))}. \end{aligned} \tag{6}$$

Inverting this approximate formula gives an equation for γ as a function of the measured traveltime t and the refraction point r.

To find r, one applies Fermat's principle of stationary time and solves

$$\frac{dt}{dr} = 0. \tag{7}$$

For the inverse problem, one solves

$$\frac{d\gamma}{dr} = 0 \tag{8}$$

to obtain the refraction point r and then evaluates the inverse formula for γ at the given r, t values. This forms the basis for an approximation algorithm to compute γ.

The background material supplies a concrete computational example. A two layer medium is given, with the upper isotropic layer of thickness 355m and velocity parameter 1030m/s, and the lower anisotropic layer of thickness 1045m, vertical wave speed of 1609m/s and anisotropic parameter $\gamma = .096$. A chart is given of various traveltimes, for X in the range 0 to 990m, and the computed gamma obtained from the inversion formula. This chart is reproduced in Table 3, including the erroneous first four traveltimes.

Offset X	Traveltime t	Inverted γ
0	1.25	N/A
90	1.25104	.0959772
190	1.27558	.0961738
290	1.3484	.0958288
390	1.02475	.0959165
490	1.04204	.0959752
590	1.06286	.0959918
690	1.08699	.0959842
790	1.11419	.0959669
890	1.1442	.0959865
990	1.17678	.0959947

Table 1: Approx. traveltime & inversion

4 Development of the solution

Our team began with an investigation aimed at understanding the elliptical velocity profile for an anisotropic material, with the intent of deriving from first principles the mathematical equations presented in the background material. We modeled the wave propagation with a constant coefficient wave equation and found the exact relationship between two orthogonal components of velocity and the anisotropic parameter γ. From this model we successfully derived all the formulas stated in the background material.

We then investigated the accuracy of the approximation used in the background work. From our model above, we produced exact formulas for Snell's law, traveltime through the two layer medium, and the inversion formula for γ. We noted that the exact formulas are no more difficult to work with than the approximations of the background material.

The next step was to redo the numerical work done using Slawinski's approximate formula, and to compare these results with the numerical results obtained using our exact formulation. We uncovered some numerical errors in the data in Table 3, and we made an observation from the numerical work that finding critical points via $dt/dr = 0$, $d\gamma/dr = 0$ is equivalent to minimizing t and γ over a permissible range of r values. We then proved this result mathematically, and used the minimization routine in MATLAB as a more robust method of solving the propagation model.

From this numerical work, we were then able to observe the sensitivity to input parameters for the γ inversion, even in the exact formulation. We noted the cause of the sensitivity, which is intrinsic to γ, and stabilized the problem by recasting the inversion in terms of two velocity parameters α and β. Plots of surfaces of intersection were created in Maple to demonstrate the stabilization.

We then produced asymptotic formulas for time of travel and γ which show more directly the sensitivity of γ, and finally proposed what we believe is a promising method of collecting detailed seismic data which gives a more stable measure of anisotropy.

5 Model, solution, and results

1 Wave equation formulation

We begin with a constant coefficient wave equation to describe the propagation of a wave in an anisotropic, two dimensional medium as

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta^2 \frac{\partial^2 u}{\partial y^2}, \tag{9}$$

where α, β are the velocity parameters in the horizontal and vertical directions, respectively. The mixed term $\frac{\partial^2 u}{\partial x \partial y}$ is absent in the equation, due to the horizontal/vertical orientation of the velocity ellipse.

For a plane wave with wave number $k > 0$ traveling at some phase angle θ_p relative to the vertical, we choose a normal vector $\mathbf{k} = (l, m) = (k \sin\theta_p, k\cos\theta_p)$ to obtain the plane wave solution

$$u = e^{i(xk\sin\theta_p + yk\cos\theta_p - \omega t)}. \tag{10}$$

Inserting u into the wave equation yields

$$\omega^2 = k^2 \left(\alpha^2 \sin^2\theta_p + \beta^2 \cos^2\theta_p\right), \tag{11}$$

from which we derive the phase velocity of the wave as

$$\begin{aligned} c_p = \frac{\omega}{k} &= \sqrt{\alpha^2 \sin^2\theta_p + \beta^2 \cos^2\theta_p} \\ &= \beta\sqrt{1 + \frac{\alpha^2 - \beta^2}{\beta^2}\sin^2\theta_p} \\ &= \beta\sqrt{1 + 2\gamma \sin^2\theta_p}, \end{aligned} \tag{12}$$

where we have identified $\gamma = (\alpha^2 - \beta^2)/2\beta^2$ as Thomsen's anisotropy parameter. Thus equation (12) captures exactly the elliptical velocity profile of the phase velocity as described in equation (1) of the background material.

Group velocity is a vector, obtained by the vector derivative

$$\mathbf{c}_g = \frac{\partial\omega}{\partial\mathbf{k}} = \left(\frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right). \tag{13}$$

As $\omega^2 = \alpha^2 l^2 + \beta^2 m^2$ we have $2\omega\frac{\partial\omega}{\partial l} = 2l\alpha^2$ and $2\omega\frac{\partial\omega}{\partial m} = 2m\beta^2$, thus

$$\mathbf{c}_g = \frac{1}{\omega}\left(\alpha^2 l, \beta^2 m\right) = \frac{k}{\omega}\left(\alpha^2 \sin\theta_p, \beta^2\cos\theta_p\right). \tag{14}$$

The magnitude of the group velocity vector is thus

$$\begin{aligned} c_g &= \frac{k}{\omega}\sqrt{\alpha^4 \sin^2\theta_p + \beta^4\cos^2\theta_p} \\ &= \frac{\sqrt{\alpha^4 \sin^2\theta_p + \beta^4\cos^2\theta_p}}{\sqrt{\alpha^2 \sin^2\theta_p + \beta^2\cos^2\theta_p}}, \end{aligned} \tag{15}$$

while the ratio of the components of $\mathbf{c}_g$ gives the tangent of the group angle θ_g as

$$\tan\theta_g = \frac{\alpha^2\sin\theta_p}{\beta^2\cos\theta_p} = \frac{\alpha^2}{\beta^2}\tan\theta_p. \tag{16}$$

Rewriting equations (15) and (16) in terms of the parameters β and γ yields immediately equations (2) and (3) of the background material, showing that our wave equation model is consistent with the description in the background material.

We now derive an exact formula for time of travel of a ray propagating through a two layer medium as in Figure 1. It is important to note that the angle θ_2 in the diagram is in fact the group angle, and the relevant velocity in the anisotropic layer is the group velocity, thus the time of travel is given by

$$\begin{aligned} t &= \frac{L_1}{c_1} + \frac{L_2}{c_g} \\ &= \frac{\sqrt{(X-r)^2 + H_1^2}}{c_1} + \sqrt{r^2 + H_2^2}\sqrt{\frac{\alpha^2\sin^2\theta_p + \beta^2\cos^2\theta_p}{\alpha^4\sin^2\theta_p + \beta^4\cos^2\theta_p}} \\ &= \frac{\sqrt{(X-r)^2 + H_1^2}}{c_1} + \sqrt{r^2 + H_2^2}\sqrt{\frac{\alpha^2\tan^2\theta_p + \beta^2}{\alpha^4\tan^2\theta_p + \beta^4}}, \end{aligned} \tag{17}$$

which reduces, by the relation $r/H_2 = \tan\theta_g = \frac{\alpha^2}{\beta^2}\tan\theta_p$ to the simple form

$$t = \frac{\sqrt{(X-r)^2 + H_1^2}}{c_1} + \sqrt{\frac{r^2}{\alpha^2} + \frac{H_2^2}{\beta^2}}. \tag{18}$$

Fermat's principle of stationary time $\frac{dt}{dr} = 0$ yields the equation

$$\frac{X-r}{c_1\sqrt{(X-r)^2 + H_1^2}} = \frac{r}{\alpha^2\sqrt{\frac{r^2}{\alpha^2} + \frac{H_2^2}{\beta^2}}} \tag{19}$$

or equivalently

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_p}{c_p} \tag{20}$$

which is Snell's law at the interface of the isotropic/anisotropic layers. Alternatively, Snell's law may be obtained by equating plane wave solutions at the two layer boundary.

Inverting equation (18) for α^2 yields

$$\alpha^2 = \frac{r^2}{(t - \sqrt{(X-r)^2 + H_1^2}/c_1)^2 - (H_2/\beta)^2} \tag{21}$$

and thus we obtain an exact formula for γ as

$$\begin{aligned}\gamma &= \frac{1}{2}\left(\frac{\alpha^2}{\beta^2} - 1\right) \\ &= \frac{1}{2}\left(\frac{r^2}{\left(t - \sqrt{(X-r)^2 + H_1^2}/c_1\right)^2 - (H_2/\beta)^2} - 1\right)\end{aligned} \tag{22}$$

We note that the exact formula for γ has a singularity in it, which is relevant when attempting to minimize γ as a function of r.

The background material mentions that the refraction point r could be obtained by solving either $dt/dr = 0$ (Fermat's principle) or $d\gamma/dt = 0$. We note the following equivalence for minima.

Proposition 4.1. *If the critical point for $dt/dr = 0$ is a local minimum, then so is the corresponding critical point for $d\gamma/dr = 0$.*

This proposition is a result of the chain rule. To summarize, observe that we have a formula $t = F(r, \gamma)$ describing traveltime in terms of the parameters r and γ; from the inversion formula, we may write γ as a function of r and t, so inserting into F gives

$$t = F(r, \gamma(r,t)). \tag{23}$$

Treating r, t as independent variables, we may differentiate (23) twice with respect to r to obtain

$$0 = \frac{\partial^2 F}{\partial r^2} + \frac{\partial^2 F}{\partial r \partial \gamma}\frac{\partial \gamma}{\partial r} + \frac{\partial F}{\partial \gamma}\frac{\partial^2 \gamma}{\partial r^2}. \tag{24}$$

But $\frac{\partial\gamma}{\partial r} = 0$ at the critical point, while $\frac{\partial^2 F}{\partial r^2}$ is positive there (as this is just the second derivative of t with respect to r, evaluated at the minimum), and $\frac{\partial F}{\partial \gamma}$ is negative for the physical reason that traveltime decreases as the anisotropy parameter increases, due to the increasing horizontal velocities. Thus from (24), the second derivative $\frac{\partial^2 \gamma}{\partial r^2}$ is positive, indicating a minimum for γ.

Since the seismic apparatus actually records the minimum time of arrival of a signal, we may assume the critical point for t, and thus for γ, is in fact minimizer. Thus we obtain the following algorithms for computing traveltime and inverting γ based on minimization.

Algorithm for traveltime

Given parameters $c_1, \alpha, \beta, H_1, H_2$ and X, minimize t with respect to r as per formula (18). Resulting t value is the traveltime.

Algorithm for γ inversion

Given parameters c_1, t, β, H_1, H_2 and X, minimize γ with respect to r as per formula (22). Resulting γ value is the anisotropic parameter.

Note that we may also minimize α^2 directly, as it is a linearly increasing function of γ.

2 Asymptotics

The numerical results show substantial errors and sensitivity to input data, in particular at the smaller values of X. We develop some asymptotic approximations using the exact equations for γ in order to obtain explicit formulas for the inverse problem. The first approximation we considered is for $X - r \ll H_1$. Expanding the exact relations (18), (19) in this approximation yields

$$t = \frac{H_1}{c_1}\left(1 + \frac{(X-r)^2}{2H_1^2}\right) + \frac{1}{\alpha\beta}\left(\alpha^2 H_2^2 + \beta^2 r^2\right)^{\frac{1}{2}}, \tag{25}$$

$$\frac{\alpha^4}{\beta^2}\frac{H_2^2}{r^2} + \alpha^2 = c_1^2 \frac{H_1^2}{(X-r)^2}. \tag{26}$$

At first sight keeping the term $(X-r)^2/H_1^2$ in (25) may seem inconsistent, since a term of the same relative order has been dropped from the right-hand side of (26). Neglecting this term in (25) facilitates a simple explicit solution to the inverse problem: r/α is obtained from (25), and its substitution into (26) yields α, which can then be eliminated. The result is

$$r(X-r) = c_1 H_1 \left[(t - H_1/c_1)^2 - H_2^2/\beta^2\right]/(t - H_1/c_1) \tag{27}$$

However, inserting traveltimes into this expression for typical parameter values results in complex r values. This is further evidence of the sensitive nature of the inversion algorithm to small errors.

Our second asymptotic formula is for almost vertical rays. Here we consider both $X, r \ll H_1$ so that the previous approximations (25), (26) hold. In addition, for almost vertical rays, the traveltime t almost equals t_0, where

$$t_0 = H_1/c_1 + H_2/\beta$$

is the traveltime in the vertical case, $X = 0$. Equations (25), (26) imply that $t - t_0, r^2$ and X^2 are of like order. We put

$$t - t_0 = \Delta^2 H_2/\beta, \quad X = \Delta\, qh, \quad r = \Delta\, ph, \tag{28}$$

where $h = (2H_1H_2c_1/\beta)^{\frac{1}{2}}$ is a convenient length scale, and p, q, Δ are scalars, with $\Delta \ll 1$. Inserting these into (25), (26), and retaining the leading order terms, gives

$$p = q - 1/q \tag{29}$$

and

$$\frac{\alpha^2}{\beta^2} = \frac{h^2}{H_2^2}\left(\frac{q^2-1}{2}\right). \tag{30}$$

The above asymptotic results for this approximation allow the identification of α/β from almost vertical traveltime measurements via simple formulas. First, Δ and q are obtained from (28) in terms of $t - t_0$ and X. Then α/β is found from (30). In terms of original parameters,

$$\frac{\alpha^2}{\beta^2} = \frac{c_1H_1}{\beta H_2}\left(\frac{X^2}{2c_1H_1(t-t_0)} - 1\right) \tag{31}$$

The accuracy of this formula is discussed further in the section on numerical results.

We may observe directly the effect of measurement errors by this vertical ray approximation of equation (30), the horizontal velocity α in the anisotropic medium is presented in terms of the measured traveltime and other parameters. Since it is available as an explicit formula, it provides an excellent resource to examine parametric dependence of the identification problem. Of particular interest is the error in α resulting from measurement error in β. Let δ be the fractional change in β. That is β is replaced by $\beta(1+\delta)$ in (31). The fractional change in α is calculated from

$$\frac{1}{\alpha^2}\Delta(\alpha^2) = \frac{1}{\alpha^2}\frac{\partial\alpha^2}{\partial\beta}\beta\delta = \left[1 - \frac{X^2}{2\alpha^2}\frac{1}{(t-t_0)^2}\right]\delta \tag{32}$$

For typical values ($\gamma = 0.06$, $X = 190$m) the right-hand side has the value -95δ. That is, a 1% error in β gives rise to a 95% error in α^2. This large amplification is, of course, due to the smallness of $t - t_0$, its occurrence in the denominator of (31), and the dependence of t_0 on β.

6 Numerical results

We implemented the minimization algorithms for finding traveltime t and anisotropic parameter γ in MATLAB, using both our exact formulas and Slawinski's approximations. The parameter values for $c_1, \gamma, \beta, H_1, H_2$, are exactly as in the computational example in the background material. We explicitly avoid the singularities in the minimization routines to guarantee accurate results.

A comparison was made of the results of the exact formula with the approximate formula, with results summarized in Table 6. The approximate values of Table 3 's are recovered (note the first four values in Table 3 are in error; Slawinski indicates an editing problem), the exact answers are in good agreement with the approximation, and our γ inversions are exact to machine accuracy. Note that in the background work, the γ inversion was not exact even for this synthetic data; we do not know if this indicates an inaccuracy in the MATHEMATICA software used in the background material, or simply the use of inexact input data in the background work.

Offset	Traveltime			Inverted γ	
X	Approx	Exact	Error	Approx	Exact
0	0.99413	0.99413	0 %	.0960000	.0960000
90	0.99579	0.99584	.01 %	.0960000	.0960000
190	1.00148	1.00171	.02 %	.0960000	.0960000
290	1.01117	1.01169	.05 %	.0960000	.0960000
390	1.02474	1.02564	.09 %	.0960000	.0960000
490	1.04204	1.04336	.13 %	.0960000	.0960000
590	1.06286	1.06465	.17 %	.0960000	.0960000
690	1.08699	1.08925	.21 %	.0960000	.0960000
790	1.11418	1.11690	.24 %	.0960000	.0960000
890	1.14420	1.14735	.28 %	.0960000	.0960000
990	1.17678	1.18034	.30 %	.0960000	.0960000

Table 2: Approximate vs. exact comparisons

We then investigated the sensitivity of the γ inversion to input data, the results of which are summarized in Table 6. Boosting t by 1 percent gave errors in γ as large as 600 percent, as did boosting β by 1 percent, with the larger errors occuring for small values of X. Even for midrange values of X, the errors are in the range of 50 to 200 percent.

However, further numerical work showed that if we compute the α parameter from the time data, the result is much less sensitive to input data, as summarized in Table 3. In this case, boosting t by 1 percent gave errors in the inverted α only as large as 43 percent (disregarding the physically impossible result of zero velocity), while boosting β by 1 percent gave errors as large as 33 percent for similar offset. Again, the larger errors appeared for small values of X, with midrange values for X giving errors on the order of 5 to 10 percent. The overall error for α is much smaller than for γ.

Offset	Time error = 1%		β error = 1%	
X	γ	Error	γ	Error
90	-.5000	621 %	-.4621	581 %
190	-.3064	419 %	-.2311	341 %
290	-.1635	270 %	-.0984	202 %
390	-.0794	183 %	-.0299	131 %
490	-.0298	131 %	.0076	92 %
590	.0008	99 %	.0298	69 %
690	.0207	78 %	.0438	54 %
790	.0343	64 %	.0532	45 %
890	.0438	54 %	.0597	38 %
990	.0509	47 %	.0644	33 %

Table 3: Parameter γ sensitivity to input error

Offset	Time error = 1%		β error = 1%	
X	α	Error	α	Error
90	0	100 %	443	75 %
190	1001	43 %	1180	33 %
290	1320	25 %	1442	18 %
390	1476	16 %	1560	11 %
490	1560	11 %	1621	8 %
590	1610	8 %	1656	6 %
690	1642	7 %	1678	5 %
790	1663	5 %	1692	4 %
890	1678	5 %	1702	3 %
990	1689	4 %	1709	3%

Table 4: Velocity α sensitivity to input error

For the same parameter values, we tested our asymptotic formulas for inverting γ, as summarized in Table 6. We see this approximation formula recovers γ from exact data to within a couple of percent.

Exact $\gamma =$	.03	.06	.09	.12	.15
X= 90	.0307	.0607	.0907	.1207	.1507
X=190	.0330	.0631	.0931	.1231	.1532
X=290	.0371	.0672	.0972	.1273	.1574

Table 5: Asymptotic results for γ

From these numerical observations, we conclude a major problem with the γ inversion is that the parameter γ is intrinsically sensitive to the input data for the given measurement technique. The problem does not arise just from the approximation originally used, or the particular numerical methods, although our minimization method in MATLAB was an improvement over the zero finding of MATHEMATICA.

A physical explanation can be given for this phenomena. In the measurement technique under study, small X values give nearly vertical rays, which measure accurately the vertical velocity parameter β, while large X values give more horizontal rays, which measure accurately the horizontal velocity parameter α. The general ray measures some weighted average of α and β. However, the anisotropic parameter $\gamma = (\alpha^2 - \beta^2)/(2\beta^2) \approx \alpha - \beta$ is the difference of the two velocity parameters, and the current measurement technique does not directly measure it. An accurate measure of γ requires very high accurate, independent measurements of α and β.

To demonstrate the stability of the α, β parameter formulation, we used MAPLE to plot certain surfaces of the parameters to show steepness of the intersection, which implied robustness of the method. The plots have been omitted from the report.

7 Further directions

In order to accurately measure the anisotropic velocity parameter γ, which is essentially the difference of the two velocity parameters α and β, it is necessary to devise a measurement technique that measures some quantity that depends directly on the difference $\alpha - \beta$. Several methods occur as possibilities, however, they are limited by the physical difficulty of performing the measurements.

A key notion is to record the differential time of travel for a single signal traveling along two different paths. If one path is near vertical, and the other more horizontal, the difference in the time of travel will be related to the difference $\alpha - \beta$. Signal processing methods can be devised that record the received signals simultaneously, and compute the differential in time of travel directly.

Generating two signal paths from one seismic event is more problematic. One possibility is to consider two different polarizations of the seismic signal, as separated out by a 3-axis geophone. Depending on the nature of the anisotropy of the geological layer, it is possible that different polarizations of a single signal will travel along different paths; the difference in time of travel will depend on $\alpha - \beta$.

Another possibility is to have more than one recording device, at widely separated positions in the bore hole. One seismic signal on the surface would generate a pulse at both receivers, again traveling along different paths. With proper placements of the receivers, it is possible to have one path nearly vertical, another more horizontal.

A third, but probably more difficult method would be to generate the seismic source at the bottom of the bore hole, and receive simultaneous signals on the surface at widely separated X offsets. Again the differential time of arrival will be a useful measure for extracting γ.

It is also quite possible that in certain geophysical applications, the precise value of γ is not directly useful, but instead the velocity parameters α and β are the important measures. In this case, the measurement technique considered in this paper is completely appropriate for a robust recovery of α from measured time of travel. The most accurate recovery of α occurs for the most horizontal ray paths; that is, for large offsets X. The minimization algorithm may be applied directly to the exact formula for α^2 for a robust computation of this parameter.

8 Conclusions

We have obtained an exact mathematical description of a geoseismic signal propagating through an anisotropic medium using a constant coefficient wave equation as the basic model. This model captures exactly the elliptical velocity profile required in the formulation of the geophysical model from which we obtained exact formulas describing the traveltime through a two layer geological structure, and an exact inversion formula for computing the anisotropic velocity parameter γ. A robust numerical method based on a minimization technique was presented as an accurate method of computing both traveltime and the inverted γ.

The exact formulas and robust numerical methods are significant improvements over the approximations and root finding methods discussed in the background material, and we note our formulation is no more difficult than these background methods.

We derived asymptotic formulas valid for the near vertical case, which describe accurately the high sensitivity of γ to the input parameters in this case. Our numerical work also confirms this sensitivity, even using exact formulas and robust numerical methods.

We conclude that the computation of the anisotropic velocity parameter γ for the given physical measurements from a series of surface signals and single borehole receiver is intrinsically unstable. By changing to the α, β velocity parameter space, we obtain an inversion method that is much less sensitive to input errors. For certain geophysical problems, the α, β parameters may suffice for an accurate description of the material.

When the anisotropic velocity parameter γ is needed directly, a different measurement technique is required. This route will require further investigation, and we have proposed a number of promising possibilities involving a differential time measure.