Customer population modelling with residence time structure

Problem presented by

Peter Grindrod

Numbercraft, Oxford

Problem statement

In many service industries, companies offer a variety of customer packages, with differing levels of service and associated charges. For example, in the case of cable companies, customers may choose to subscribe to different bundles of channels and may also buy phone and internet services. From time to time, customers will upgrade to a more expensive package, or possibly downgrade or discontinue their contract altogether. Numbercraft asked the Study Group to consider models for how the number of customers on each type of contract will change over time. Such models could be used to forecast companies' future income and also to ensure that marketing campaigns have maximum impact.

Study Group contributors

J. N. Dewynne (University of Oxford)
E. J. Hinch (University of Cambridge)
S. D. Howison (University of Oxford)
D. F. Parker (University of Edinburgh)
R. E. Wilson (University of Bristol)

Report prepared by

Eddie Wilson (University of Bristol)

1 Introduction

This problem, brought by Prof. Peter Grindrod of Numbercraft[1], was concerned with the population modelling of customers of *e.g.* telecommunications companies. Such companies offer a variety of service packages of differing expense. (For example, in the case of cable companies, customers may choose to subscribe to different bundles of channels and may also buy phone and internet services.) From time to time, customers will upgrade to a more expensive package, or possibly downgrade or discontinue their contract altogether. Numbercraft would like to predict how the number of customers on each type of contract will change, so that they may forecast companies' future income, and also so that they may advise marketing campaigns with maximum timeliness and effect.

The idealised situation is shown in Figure 1. We suppose that there are n discrete classes of customer account. In addition, we may have a removed class which corresponds to customers who discontinue their contracts and from whom no income is derived. We aim to model the populations and flows between the different customer classes. In the simplest case, this gives rise to a simple system of ordinary differential equations.

However here we add an extra level of sophistication: we allow customers' behaviour, specifically their rates of switching class, to be a function of their residence times in those classes. For example, in the case of a cable company, after a user has experienced a satisfactory period on the basic package, they may become increasingly likely to adopt extra channels and services. A second effect that we might model with this approach is that of lock-in, where customers sign up for a package for a prescribed minimum period.

Clearly it is no longer sufficient to keep track of the total number of customers $u_i(t)$ in each class i. Rather, we require variables that describe the residence time structure of each population. We use $u_i(s,t)$ to denote the density of individuals in class i, with residence time $s \geq 0$, at time t. The Study Group was asked to develop a model under the following assumptions:

- At any time, the rate density of individuals changing class does not depend on the populations of destination classes. In fact, we suppose that the total rate of customers leaving class i is a linear functional of the population of class i itself.
- When individuals change class, they commence in their new class with residence time zero.

2 Model derivation

We define $k_i(s)$ to be the rate density function for customers leaving class i. By this, we mean that in a small time interval δt, a total of $k_i(s)u_i(s,t)(\delta t)(\delta s)$ individuals with residence time between s and $s + \delta s$ (δs small) will leave class i. It follows that

$$\frac{\partial u_i}{\partial s} + \frac{\partial u_i}{\partial t} = -k_i(s)u_i, \qquad s > 0, \tag{1}$$

[1] http://www.numbercraft.co.uk/

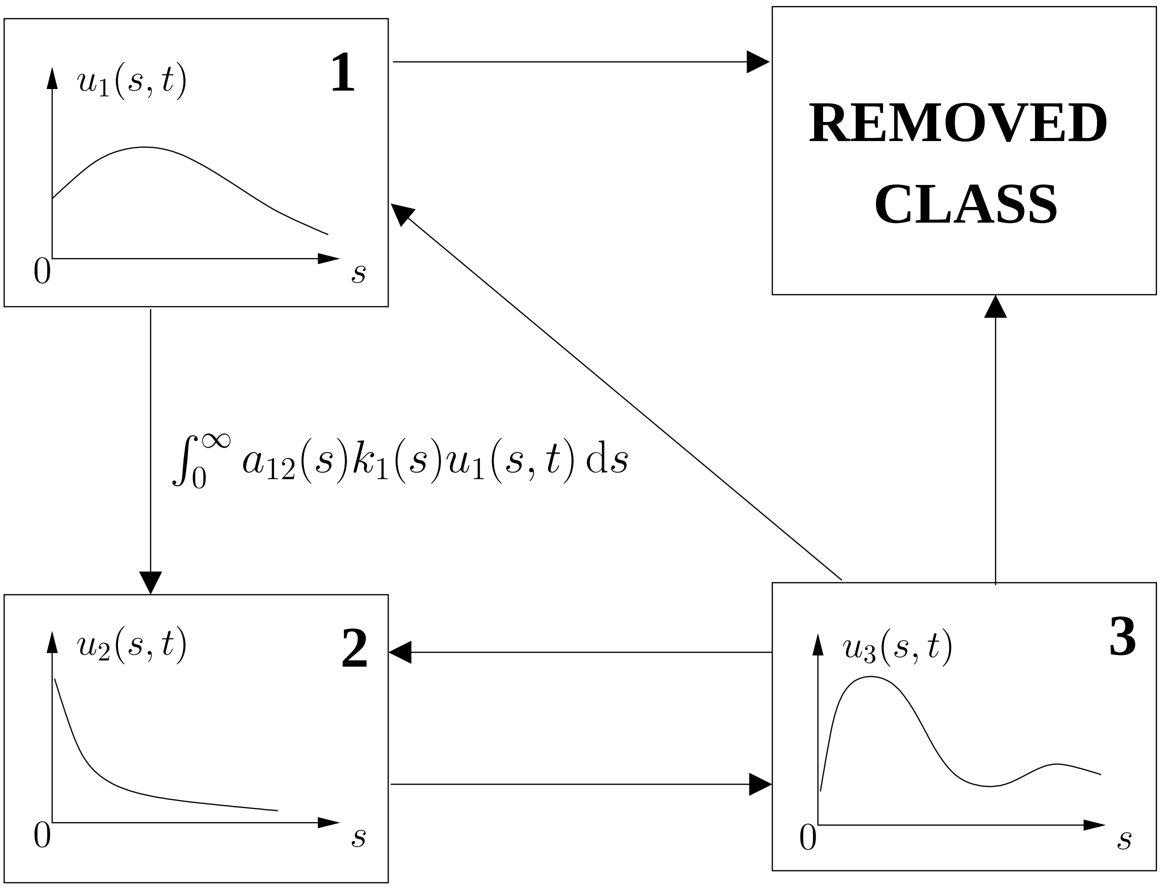

Figure 1: An example of a compartment model of customer populations, which includes residence time s structure. In this case there are $n = 3$ classes of customer account, in addition to a removed class. Possible movements between account types are indicated by arrows, and a model formula for the flow rate from account class 1 to account class 2 is given.

or in vector form

$$\frac{\partial \mathbf{u}}{\partial s} + \frac{\partial \mathbf{u}}{\partial t} = -\mathbf{K}(s)\mathbf{u}, \qquad s > 0, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{K} = \mathrm{diag}(k_1, k_2, \ldots, k_n)$. Here the differential operator on the left hand side states that individuals who do not change class age at rate one.

It is now necessary to consider the destination of those customers who change class. To this end, we introduce a matrix $\mathbf{A}(s)$ with entries $a_{ij}(s)$ which describes how the flow leaving class i with residence time s is split between the other classes j. Since the key quantity is $\mathbf{A}^{\mathrm{T}}\mathbf{K}$, we normalise $\mathbf{A}^{\mathrm{T}}$ so that each row sums to one, *i.e.* $\mathbf{A}$ is a Markov matrix.

It follows that, at time t, the total flow rate of customers from class i to class j is $\int_0^\infty a_{ij}(s)k_i(s)u_i(s,t)\,\mathrm{d}s$. When contributions from all customers of all residence times are considered, we have that the total flow into class i is equal to the sum of contributions from all classes j together with a source term for fresh customers, *i.e.*

$$\sum_{j=1}^{n} \int_{0}^{\infty} a_{ji}(s) k_j(s) u_j(s,t)\,\mathrm{d}s \; + \; g_i(t), \tag{3}$$

from which we may obtain the $s = 0$ boundary condition,

$$u_i(0,t) \quad = \quad \int_{0}^{\infty} \sum_{j=1}^{n} a_{ji}(s) k_j(s) u_j(s,t)\,\mathrm{d}s \; + \; g_i(t), \tag{4}$$

or in vector form

$$\mathbf{u}(0,t) \quad = \quad \int_{0}^{\infty} \mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\mathbf{u}(s,t)\,\mathrm{d}s \; + \; \mathbf{g}(t). \tag{5}$$

Equations (2) and (5) together define the hyperbolic PDE and boundary conditions which we consider in the remainder of this report.

3 Initial data

We now consider what initial (in t) data should be prescribed. We take the view that $t = 0$ should correspond to an *absolute zero* time prior to which there are no customers in the system. It follows from this assumption that any customers starting in the system at $t = 0$ must do so with residence time $s = 0$. Thus we may take initial data

$$\mathbf{u}(s,0) = \delta(s)\mathbf{u}_0, \tag{6}$$

where $\mathbf{u}_0 \geq \mathbf{0}$ is the vector of initial populations at $s = 0$, $t = 0$, and δ is the usual distribution.

4 Singular solution component

We now consider the solution of the PDE IBVP defined by (2), (5) and (6). First note that (2) may be re-written as an ODE

$$\mathbf{u}_\tau = -\mathbf{K}\mathbf{u} \tag{7}$$

along the characteristics of the system. Here $\tau = s + t$. Further the residence time of customers in the system is less than or equal to the age of the system, so $s \leq t$, and also $s \geq 0$, $t \geq 0$. Figure 2 depicts the wedge of the s–t plane in which the solution is defined, as well as the characteristics $t - s =$ constant ≥ 0.

The key to the solution is to note that the distributional initial data (6) will propagate along the boundary characteristic $s = t$ as a singular distributional solution component, although it will continuously lose mass via (2) and (5) to the $s = 0$, $t > 0$ line: other $t - s =$ constant > 0 characteristics then propagate this mass forward into the interior of the wedge-shaped domain. However, the components of the solution along these

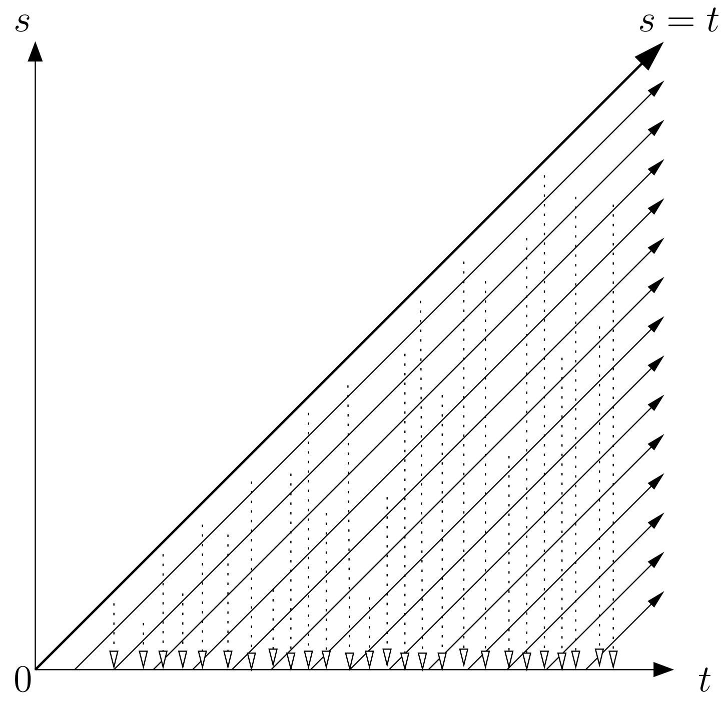

Figure 2: The s–t solution plane and domain of definition, when distributional data is provided at $s = t = 0$. Characteristics take the form $t - s = \text{constant} \geq 0$ and are drawn as solid lines. A distributional solution component propagates along $s = t$. The continuous feedback of all parts of the solution to the $s = 0$ boundary is indicated by dotted lines.

characteristics will also lose mass to the $s = 0$, $t > 0$ line in a continuous way, from which new characteristics will fill out the wedge, and so on.

We now write

$$\mathbf{u}(s,t) = \mathbf{u}_{\text{sing}}(s,t) + \tilde{\mathbf{u}}(s,t), \tag{8}$$

where $\mathbf{u}_{\text{sing}}$ is the singular solution component propagating along $s = t$ and $\tilde{\mathbf{u}}(s,t)$ is the regular solution component in the interior of the wedge.

Combining (6) with the ODE (7) yields

$$\mathbf{u}_{\text{sing}}(s,t) = \delta(t-s)\mathbf{E}(s)\mathbf{u}_0, \tag{9}$$

where

$$\mathbf{E}(s) := \exp\left(-\int_0^s \mathbf{K}(\sigma)\,\mathrm{d}\sigma\right), \tag{10}$$

is the evolution operator along characteristics. Note that

$$\mathbf{r}(s) := -\mathbf{E}'(s) = \mathbf{K}(s)\mathbf{E}(s) \tag{11}$$

gives the vector of *switching time* density functions, *i.e.* the probability density functions of individuals leaving their classes at time $s > 0$.

5 Regular solution component

We now proceed from (8) by noting that since $\mathbf{u}_{\mathrm{sing}}$ satisfies (7), we also have

$$\tilde{\mathbf{u}}_\tau = -\mathbf{K}\tilde{\mathbf{u}}. \tag{12}$$

Thus for $s < t$, we have

$$\tilde{\mathbf{u}}(s,t) = \mathbf{E}(s)\mathbf{u}(0, t-s), \tag{13}$$

using the evolution along characteristics. We now substitute (8) and (9) into the boundary conditions (5), so that

$$\tilde{\mathbf{u}}(0,t) + \delta(t)\mathbf{E}(0)\mathbf{u}_0 \;=\; \int_0^\infty \mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\left[\tilde{\mathbf{u}}(s,t) + \delta(t-s)\mathbf{E}(s)\mathbf{u}_0\right] \mathrm{d}s \;+\; \mathbf{g}(t), \tag{14}$$

and so for $t > 0$

$$\tilde{\mathbf{u}}(0,t) \;=\; \int_0^t \mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\tilde{\mathbf{u}}(s,t)\,\mathrm{d}s \;+\; \mathbf{A}^{\mathrm{T}}(t)\mathbf{K}(t)\mathbf{E}(t)\mathbf{u}_0 \;+\; \mathbf{g}(t). \tag{15}$$

Note that the range of integration is truncated since all solution components are zero for $s > t$. We may now substitute (13) into (15) to give

$$\tilde{\mathbf{u}}(0,t) \;=\; \int_0^t \mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\mathbf{E}(s)\tilde{\mathbf{u}}(0,t-s)\,\mathrm{d}s \;+\; \mathbf{A}^{\mathrm{T}}(t)\mathbf{K}(t)\mathbf{E}(t)\mathbf{u}_0 \;+\; \mathbf{g}(t). \tag{16}$$

We have thus reduced the search for the regular solution component to the solution of a Volterra integral equation for the solution's $s = 0$ boundary data. The distributional initial data contributes a non-homogeneous term which decays under mild conditions on the switching rate density functions $k_i(s)$.

Note that once boundary data $\tilde{\mathbf{u}}(0,t)$ has been found, integration along characteristics gives the solution $\tilde{\mathbf{u}}(s,t)$, $s > 0$, in the full wedge-shaped domain.

6 Solution by Laplace transforms

Let us use the short-hand $\mathbf{f}(t) := \tilde{\mathbf{u}}(0,t)$. The Laplace transform of (16) yields

$$\mathcal{L}(\mathbf{f}) = \mathcal{L}(\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{E})\mathcal{L}(\mathbf{f}) + \mathcal{L}(\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{E})\mathbf{u}_0 + \mathcal{L}(\mathbf{g}), \tag{17}$$

so that

$$\mathcal{L}(\mathbf{f}) = \left[\mathbf{I} - \mathcal{L}(\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{E})\right]^{-1}\left[\mathcal{L}(\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{E})\mathbf{u}_0 + \mathcal{L}(\mathbf{g})\right], \tag{18}$$

and in principle $\mathbf{f}$ may be found by the inverse transform. This process is tractable when

$$\mathbf{B}(t) := \mathbf{A}^{\mathrm{T}}(t)\mathbf{K}(t)\mathbf{E}(t) \tag{19}$$

is purely exponential in time. From (10), we observe that this is only possible if the switching rate density function is independent of residence time, and provided $\mathbf{A}(t) = \exp(-t\boldsymbol{\Lambda})\mathbf{A}_0$, *i.e.* provided the splitting matrix is purely exponential in time.

(Since elements of $\mathbf{A}$ must be non-negative for all $t \geq 0$, it follows that all elements of diagonal $\mathbf{\Lambda}$ have non-negative real parts, and all elements of $\mathbf{A}_0$ are non-negative. Further, since $\mathbf{A}$ is a Markov matrix, 0 is automatically an eigenvalue.)

Next note that $\mathbf{K}$ and $\exp(-t\mathbf{\Lambda})$ are diagonal and hence commute, so that under the above conditions

$$\mathbf{B}(t) = \mathbf{A}_0^{\mathrm{T}}\mathbf{K}\exp\left[-t(\mathbf{K}+\mathbf{\Lambda})\right], \tag{20}$$

and thus

$$\mathcal{L}(\mathbf{B}) = \mathbf{A}_0^{\mathrm{T}}\mathbf{K}(\mathbf{K}+\mathbf{\Lambda}+p\mathbf{I})^{-1}. \tag{21}$$

Thus (18) yields

$$\mathcal{L}(\mathbf{f}) = \left[\mathbf{I} - \mathbf{A}_0^{\mathrm{T}}\mathbf{K}(\mathbf{K}+\mathbf{\Lambda}+p\mathbf{I})^{-1}\right]^{-1}\left[\mathbf{A}_0^{\mathrm{T}}\mathbf{K}(\mathbf{K}+\mathbf{\Lambda}+p\mathbf{I})^{-1}\mathbf{u}_0 + \mathcal{L}(\mathbf{g})\right] \tag{22}$$

$$= \left[\left(\mathbf{I} - \mathbf{A}_0^{\mathrm{T}}\right)\mathbf{K}+\mathbf{\Lambda}+p\mathbf{I}\right]^{-1}\left[\mathbf{K}+\mathbf{\Lambda}+p\mathbf{I}\right]\left[\mathbf{A}_0^{\mathrm{T}}\mathbf{K}(\mathbf{K}+\mathbf{\Lambda}+p\mathbf{I})^{-1}\mathbf{u}_0 + \mathcal{L}(\mathbf{g})\right]. \tag{23}$$

Only the first term generates poles, and so we may observe that the boundary values of the solution are exponential in t with exponents given by the eigenvalues of $(\mathbf{A}_0^{\mathrm{T}}-\mathbf{I})\mathbf{K}-\mathbf{\Lambda}$. Since $\mathbf{A}_0^{\mathrm{T}}$ row sums to one, and $\mathbf{K}$, $\mathbf{\Lambda}$ are non-negative diagonal matrices, Gerschgorin's theorem tells us that the eigenvalues have non-positive real parts, and exponential decay to a steady state is the only possible dynamics.

The above procedure is rather disappointing as it does not give explicit progress in the case where the switching rate density $\mathbf{K}$ has non-trivial dependence on residence time s. However, explicit progress can be made if $\mathbf{K}$ is constant, but the splitting matrix $\mathbf{A}$ is time periodic (*i.e.* all entries of $\mathbf{\Lambda}$ are purely imaginary or zero), which might be used to model seasonal variation in customer behaviour.

Of course, it may well be the case that evaluating and inverting Laplace transforms numerically gives an efficient computational procedure for solving (16): however, we have not investigated this idea in detail.

7 Steady states

Steady state solutions $\mathbf{u}^* \geq \mathbf{0}$ of (16) might be recovered if we set sources $\mathbf{g} = \mathbf{0}$, and if the limit $t \to \infty$ is taken, so that the inhomogeneous term due to initial data $\mathbf{u}_0$ is forgotten by the system. This loss of memory occurs if all k_i are strictly positive and bounded away from zero, so that $\mathbf{E}$ decays exponentially or faster. Clearly then

$$\mathbf{u}^* = \left[\int_0^\infty \mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\mathbf{E}(s)\,\mathrm{d}s\right]\mathbf{u}^*, \tag{24}$$

and thus a necessary condition for such a solution is that $\mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\mathbf{E}(s) \in \mathrm{L}^1(0,\infty)$. Since $\mathbf{A}$ is bounded, this can be guaranteed if all k_i are strictly positive, bounded away from zero, and bounded from above, so that $\mathbf{K}(s)\mathbf{E}(s)$ decays exponentially.

8 Some constant coefficient examples

Under the conditions on switching rate densities described in the previous section, the dynamics of (16), and consequently the dynamics of (2), (5) are rather uninteresting, as equilibration to a steady state is inevitable. In this section, we examine the time scale of this equilibration. Further, we examine whether equilibration is necessarily monotone, or whether it can contain an oscillatory component. For reasons of simplicity, we suppose that the splitting matrix $\mathbf{A}$ and switching rate density $\mathbf{K}$ are constant. Thus, by (23), solutions are exponential in time with exponents given by the eigenvalues of $(\mathbf{A}^{\mathrm{T}} - \mathbf{I})\mathbf{K}$.

Example 1. We consider the scalar case $n = 1$. This might model the population holding a certain credit card, and having one's residence time reset to zero corresponds to being issued with a replacement card, either due to theft or to expiry. We have $\mathbf{A}^{\mathrm{T}} - \mathbf{I} = (0)$, so that boundary data $u(0, t)$ is steady for $t > 0$.

Using the theory of Sections 4, 5 and 6 we may construct the explicit solution

$$u(s,t) = \left[k \exp(-ks)\chi_{[0,t)}(s) + \delta(t)\exp(-kt)\right] u_0, \tag{25}$$

where χ is the indicator function defined by

$$\chi_{[0,t)}(s) := \begin{cases} 1 & 0 \le s < t, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Thus the initial cluster of customers ages at rate one, whilst losing mass, and sweeps out the eventual steady profile $u(s) = k\exp(-ks)u_0$ behind it.

Example 2. We consider a system with $n = 2$ components, and we attempt to generate oscillatory decay by using the splitting matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{27}$$

which 'cycles customers' between the two states. In this case $(\mathbf{A}^{\mathrm{T}} - \mathbf{I})\mathbf{K}$ has eigenvalues of 0 (corresponding to the steady state) and $-(k_1 + k_2)$, which is real, and so only monotone decay is possible. Interestingly, the decay rate is the sum of the individual switching rate components, *i.e.* the system relaxes faster than one might expect.

Example 3. We now consider a system with $n = 3$ components, and we use the splitting matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{28}$$

Thus

$$(\mathbf{A}^{\mathrm{T}} - \mathbf{I})\mathbf{K} = \begin{pmatrix} -k_1 & +k_2 & 0 \\ 0 & -k_2 & +k_3 \\ +k_1 & 0 & -k_3 \end{pmatrix}. \tag{29}$$

As always, we find an eigenvalue $\mu = 0$, in addition to two more, which satisfy

$$\mu^2 + (k_1 + k_2 + k_3)\mu + (k_1k_2 + k_1k_3 + k_2k_3) = 0. \tag{30}$$

Two extreme cases are worth mention: (i) $0 < k_2,\ k_3 \ll k_1$, in which case we obtain a pair of real eigenvalues (and hence monotone decay); and (ii) $k_1 = k_2 = k_3$, in which case we obtain eigenvalues with non-zero imaginary parts, so that oscillatory decay occurs.

9 Piecewise constant functions

Given our explicit progress with constant switching rate densities and constant splitting coefficients, the next possibility (which we have not investigated fully) would appear to be the use of piecewise constant switching rates. A first investigation might involve

$$\mathbf{K}(s) = \begin{cases} \mathbf{K}_\alpha & 0 < s \le s^*, \\ \mathbf{K}_\beta & s > s^*, \end{cases} \tag{31}$$

where $\mathbf{K}_\alpha$, $\mathbf{K}_\beta$ and s^* are constant. In this case it is readily shown that

$$\mathbf{E}(s) = \begin{cases} \exp\left(-s\mathbf{K}_\alpha\right) & 0 < s \le s^*, \\ \exp\left(+s^*\left(\mathbf{K}_\beta - \mathbf{K}_\alpha\right)\right)\exp\left(-s\mathbf{K}_\beta\right) & s > s^*, \end{cases} \tag{32}$$

and the Laplace transform of $\mathbf{A}^{\mathrm{T}}(s)\mathbf{K}(s)\mathbf{E}(s)$ may be calculated explicitly if $\mathbf{A}$ is purely exponential and if the range of integration is partitioned at $s = s^*$.

We might use the above approach to model lock-in periods by setting some components of $\mathbf{K}_\alpha$ to zero. However, zero components of $\mathbf{K}_\beta$ correspond to absorbing classes (effectively removed classes) which customers do not leave after s^*. If the graph whose edges are defined by the non-zero elements of $\mathbf{A}$ is well-connected, then all customers will end up in an absorbing class as $t \to \infty$. If there is more than one such absorbing class, then there will be non-unique steady states as $t \to \infty$.

10 Distributional coefficients

In all cases considered so far, solutions have decayed to steady states as $t \to \infty$. However, this need not be the case if we allow distributional switching rate density functions. For example, if we take a two component system with splitting matrix given by (27) and use switching rate density functions

$$k_1(s) = \delta(s_1) \qquad \text{and} \qquad k_2(s) = \delta(s_2), \tag{33}$$

where s_1, $s_2 > 0$, then we will obtain solutions of period $s_1 + s_2$ where customers are cycled between the two classes.

An interesting conjecture is that regular switching rate density functions can never produce this periodic behaviour: most likely this conjecture can be proven by deriving asymptotics for the poles of the Laplace transform (18) of boundary data.

Interesting analysis could be carried out on regular, but very spikey (‘almost δ’) switching rate density functions. We would expect to see something like periodic behaviour, but which must decay to a steady state as $t \to \infty$. Most likely the solutions will exhibit such decay only over very long time scales, which might also be obtained from asymptotics of the Laplace transform.

11 Nonlinear model extensions[2]

An interesting extension to this modelling would be to consider switching rate density functions which depended on customer populations themselves. For example, word-of-mouth advertising from satisfied customers of a particular package may result in a very rapid take up of that package by others. Similarly, a company may not be able to service adequately customers of an over-subscribed package, which may lead to a large switching rate out of that account class.

Note that only the right hand side of PDE (2) is changed, so that characteristics remain $t - s =$ constant. However, integration along characteristics now involves solving nonlinear (albeit scalar) ODEs. Furthermore, the boundary feedback (5) will also become nonlinear, although in principle one can produce a *nonlinear* integral equation analagous to (16) which also involves the evolution operator $\mathbf{E}(t)$ of the *nonlinear* ODE problem. One would have to resort to numerics, and it would seem advisable to have specific examples in mind before proceeding.

12 User variability

We now add an extra level of complexity to the modelling, by supposing that not only are there several discrete classes of customer account, but also that there are different types of individual customer with different switching rate density functions.

One modelling approach is to suppose that there are m discrete types of customer, and since all interactions are linear, we may thus create m disjoint copies of the PDE system discussed earlier.

Similarly, we might suppose that customer type is parametrised by a continuous variable θ (or a collection of continuous variables θ). We then have density functions $\mathbf{u}(s, t, \theta)$, switching rate densities $\mathbf{K}(s, \theta)$ and splitting matrices $\mathbf{A}(s, \theta)$. However, the basic PDE model (2) and boundary conditions (5) are unaltered as there is as yet no mechanism for producing derivatives such as $\partial\mathbf{u}/\partial\theta$. It is possible to attempt an averaging over θ; this will lead to models whose modified coefficients contain θ derivatives of $\mathbf{K}(s, \theta)$ and $\mathbf{A}(s, \theta)$. No new dynamics are introduced by this idea.

However, if we suppose that a customer's type itself changes slowly in time, our models contain Taylor diffusion. This may be seen formally if we consider a conservation law for class i of customers of type θ (assumed scalar for simplicity) at time t and residence time s. We may write

$$\begin{aligned} &u_i(s, t + \delta t, \theta) = (1 - k_i \delta t) u_i(s - \delta t, t - \delta t, \theta) \\ &+ \alpha \delta t \left[u_i(s - \delta t, t - \delta t, \theta + \delta\theta) - 2u_i(s - \delta t, t - \delta t, \theta) + u_i(s - \delta t, t - \delta t, \theta - \delta\theta) \right], \end{aligned} \tag{34}$$

where the second line of terms models slow drift in θ. Expanding under appropriate scaling yields the modified PDE

$$\frac{\partial u_i}{\partial s} + \frac{\partial u_i}{\partial t} = -k_i u_i + \sigma^2 \frac{\partial^2 u_i}{\partial \theta^2}. \tag{35}$$

[2]Some of the ideas in this section were investigated briefly by David Parker (Edinburgh) and Jeff Dewynne (OCIAM).

We have yet to analyse this type of model in detail.

13 Future work

The next step of this work should involve a numerical solver for the Volterra integral equation (16). Unfortunately, solving crudely up to time $t_{\max}$ will take $O(t_{\max}^2)$ operations, as at each time step an integral from 0 to t must be performed. This is unsurprising considering that we are effectively solving a PDE on the two-dimensional wedge of Figure 2. However for large t, the contributions to the integral for sufficiently large s are small, if $\mathbf{E}(s)$ is exponentially decaying. It is possible that a smart remeshing scheme can take advantage of this fact by using a coarser mesh for large s and so saving operations.

A second possibility for numerical work is that one attempt the calculation and inversion of the Laplace transform (18) numerically.

Finally, one should perhaps consider some real examples of customer account structure, preferably with data on switching rate densities, and try to fit some of the models discussed throughout this report.