Enhanced Leak Detection

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Section 1: Introduction

A key requirement for Veeder-Root's Enhanced Leak Detection System is that it be able to test *in situ* for the presence of leaks at gasoline dispensing facilities. Aside from the obvious issues of safety and lost product, this functionality is obligatory for compliance with environmental standards mandated by federal and state oversight bodies, such as the California State Water Resources Control Board (SWRCB). The SWRCB demands a testing procedure that includes conditions as close to operational as possible, while still using environmentally safe gases as a test fluid. Although the test parameters (e.g., pressure) are allowed to deviate from operating conditions in order to facilitate the test procedure, a prescribed rescaling of the test thresholds must then be applied to account for the deviation. Whether the test is run at operation conditions or in a slightly different parameter regime, the fact that the testing must be done on the product and return lines after installation at a service station presents significant challenges in devising an effective test strategy.

The above-mentioned group of researchers, including university professors, postdoctoral fellows, graduate students, and scientists from Veeder-Root, was presented with the challenge of determining a suitable procedure that would address the issue of *in-situ* testing in an implementable and cost-effective way. This report documents the results of that analysis, performed as part of the Workshop on Mathematical Problems in Industry, held at Worcester Polytechnic Institute in Massachusetts from June 13 to 17, 2005.

As described in Section 2, the immediate challenge presented by the in-situ testing requirement is that the leak is almost invisible relative to environmental sources of noise, particularly from temperature fluctuations at the level of the tank due to heating at ground level (refer to Section 5 for analysis). Section 3 proposes a particular measurement strategy that provides a signal-to-noise ratio high enough to address in part the issue of false positives and false negatives caused by temperature fluctuations. Section 4 considers the possibility of actively forcing the primary and secondary spaces at high frequency to further mitigate the problem of environmental noise, and Section 6 analyzes the conditions under which this forcing produces a detectable leak rather than a spurious response due to pipe flexure. Sections 7 and 8 discuss methods to identify and locate leaks based either on flushing tracer gases from the secondary space or on sending an acoustic wave through the secondary space and measuring the response. Finally, the conclusions and recommendations for further study are given in Section 9.
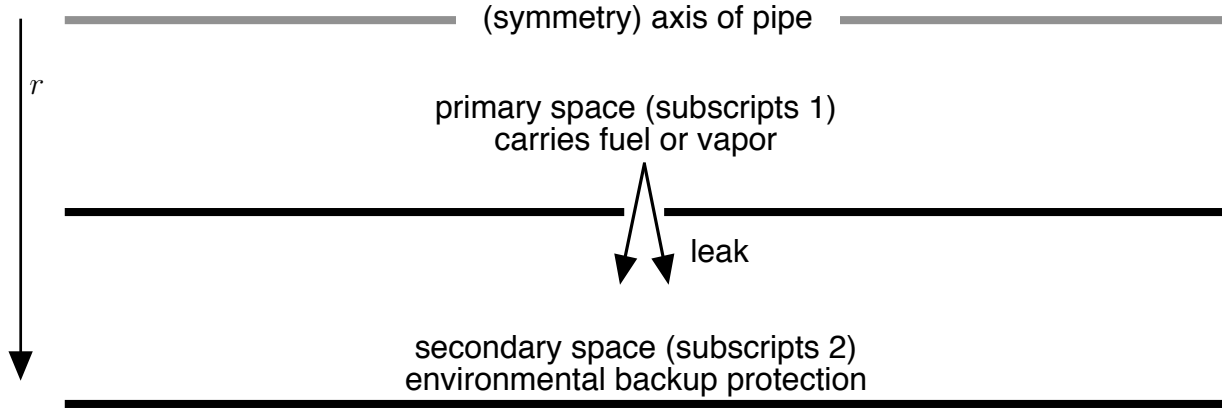
Section 2: Estimating Parameters

Figure 2.1. Schematic of pipeline.

We wish to model the amount of leakage caused by a hole in a pipeline, which is really the interior cylinder of a double-wall design (see Fig. 2.1). Gas is injected at some known pressure into the *primary space* (*i.e.*, that where the fuel or vapor would normally travel). Pressure sensors are embedded there, as well as in the *secondary space* (*i.e.*, the area used to keep leaks from penetrating into the environment).

First, we wish to construct order-of-magnitude estimates for some of the parameters in our problem. We begin by considering the gas injected during the experiment to be an ideal gas, so the following law applies:

$$\tilde{P}V = \tilde{N}R\tilde{T}, \tag{2.1}$$

where the tildes represent variables with dimensions that will later be normalized. Here $\tilde{P}$ is the pressure, V is the volume, $\tilde{N}$ is the number of moles of the gas, R is the ideal gas constant, and $\tilde{T}$ is the temperature. (See Appendix B for a discussion of why the ideal gas law is an adequate approximation.)

The first quantity we wish to estimate is the size of fluctuations occurring in the pressure in the secondary space due to daily changes in temperature. Then we can compare the size of these fluctuations to those from any potential leak. Hence we rearrange (2.1) and take the derivative with respect to $\tilde{t}$ to obtain

$$\frac{\tilde{P}_2}{\tilde{T}} = \frac{\tilde{N}_2 R}{V_2} \tag{2.2a}$$

$$\tilde{p}_2^T \equiv \left(\frac{d\tilde{P}_2}{d\tilde{t}} \right)_{\tilde{N}_2 \text{ fixed}} = \frac{\tilde{N}_2 R}{V_2} \frac{d\tilde{T}}{d\tilde{t}}, \tag{2.2b}$$

where the subscript "2" refers to the secondary space and the superscript "T" indicates we are tracking only those changes due to temperature variations. (Here we have assumed that the temperature is the same in both spaces.)

Note that since we are tracking fluctuations due only to the temperature, we treat $\tilde{N}_2$ as a constant. The system doesn't include concentration sensors. Thus, it is useful to eliminate $\tilde{N}_2$ from the analysis by noting from (2.2a) that the fraction in (2.2b) can be replaced (as an average used for estimation purposes only) by $\bar{P}_2/\bar{T}$, where the bar denotes an average or characteristic value. Thus we have

$$\tilde{p}_2^T = \frac{\bar{P}_2}{\bar{T}} \frac{d\tilde{T}}{d\tilde{t}}. \tag{2.3}$$

where we have used (2.1).

To obtain the most conservative estimate for our system, we want to calculate the largest possible value of $\bar{p}_2^T$ (a typical value of the pressure fluctuation). From the presentation slides we have that a maximal average temperature fluctuation in the fuel tank (not the pipe) would be

$$\frac{d\tilde{T}}{d\tilde{t}} = 0.5 \, \frac{\text{K}}{\text{hr}}. \tag{2.4}$$

Computing an upper bound for $\bar{p}_2^T$ requires that we use the lower bound in (A.6) and set $\bar{P}_2$ to atmospheric pressure. Thus we have

$$\bar{p}_2^T = \frac{1.01 \times 10^5 \text{ Pa}}{283 \text{ K}} \left(0.5 \, \frac{\text{K}}{\text{hr}} \right) = 1.78 \times 10^2 \, \frac{\text{Pa}}{\text{hr}}. \tag{2.5}$$

For the leak calculations, rearrange (2.1) differently before taking the derivative:

$$\tilde{p}_2^l \equiv \left(\frac{d\tilde{P}_2}{d\tilde{t}} \right)_{\tilde{T} \text{ fixed}} = \frac{\bar{T}R}{V_2} \frac{d\tilde{N}_2}{d\tilde{t}}, \tag{2.6}$$

where the superscript "l" indicates we are tracking only those changes due to the leak. We again work to avoid $\tilde{N}_2$ by noting that the maximum leak tolerance is given in units of volume per hour at standard operating pressure, which is atmospheric, so we have

$$P_\text{a} \frac{d\tilde{V}_l}{d\tilde{t}} = R\bar{T} \frac{d\tilde{N}_2}{d\tilde{t}} \tag{2.7a}$$

$$\tilde{p}_2^l = \frac{P_\text{a}}{V_2} \frac{d\tilde{V}_l}{d\tilde{t}}. \tag{2.7b}$$

To get an estimate of the variation due to the leak (which we denote by a bar), we substitute the value of V_2 in (A.4), (A.1), and of course (A.7):

$$\bar{p}_2^l = \frac{1.01 \times 10^5 \text{ Pa}}{110 \text{ gal}} \left(5 \times 10^{-3} \, \frac{\text{gal}}{\text{hr}} \right) = 4.61 \, \frac{\text{Pa}}{\text{hr}}. \tag{2.8}$$

Since this value is quite small, we must ensure that the pressure sensors commercially available would indeed be able to measure such a small leak. The differential pressure ($\tilde{p}$) sensor currently in use has a range of 10 inches of water and has a drift rate of 0.04% of that per hour. This calculation is based on four error sources, including a temperature differential of $2°$ F/hr, changes in humidity, and short-term (24 hour) stability. In metric units, this translates into a drift of

$$\frac{4 \times 10^{-4}}{\text{hr}}(10 \text{ in } H_2O)\left(\frac{249 \text{ Pa}}{\text{in } H_2O}\right) = 9.96 \times 10^{-1} \frac{\text{Pa}}{\text{hr}}. \tag{2.9}$$

Then to compute the ratio, we take the quotient of (2.8) and (2.5):

$$\frac{\bar{p}_2^{\text{l}}}{\bar{p}_2^{T}} = \frac{4.61 \text{ Pa/hr}}{1.78 \times 10^2 \text{ Pa/hr}} = 2.58 \times 10^{-2}. \tag{2.10}$$

Therefore, the pressure variations caused by the leak are more than an order of magnitude smaller than those caused by temperature variations. Thus we cannot discern them by amplitude alone—rather, we will have to be more clever to design experiments where we can measure the leak pressure change.

Section 3: Using Pressure Rates

Unfortunately, the pressure meters currently in use are not very accurate: they can measure absolute pressure up to about only 5% accuracy. On the other hand, the finer meters are quite accurate, exhibiting the drift quoted in section 2. Thus, it is desirable to derive a formula in which $\tilde{p}$ figures prominently. (Though technically it is $\Delta \tilde{P}$ which is measured, since we need to calculate rates eventually, we deal with $\tilde{p}$ instead.)

At the beginning we take into account the fact that $\tilde{T}$ may vary. Taking the derivative of (2.1) with respect to $\tilde{t}$ in each region, we have

$$\tilde{p}_j V_j = R \left(\tilde{T}_j \frac{d\tilde{N}_j}{d\tilde{t}} + \tilde{N}_j \frac{d\tilde{T}_j}{d\tilde{t}} \right) \tag{3.1a}$$

$$\frac{\tilde{p}_j}{\bar{P}_j} = R \left(\frac{\tilde{T}_j}{V_j \bar{P}_j} \frac{d\tilde{N}_j}{d\tilde{t}} + \frac{\tilde{N}_j}{V_j \bar{P}_j} \frac{d\tilde{T}_j}{d\tilde{t}} \right)$$

$$\frac{\tilde{p}_2}{\bar{P}_2} - \frac{\tilde{p}_1}{\bar{P}_1} = \frac{R\tilde{T}_2}{V_2 \bar{P}_2} \frac{d\tilde{N}_2}{d\tilde{t}} + \frac{1}{\tilde{T}_2} \frac{d\tilde{T}_2}{d\tilde{t}} - \frac{R\tilde{T}_1}{V_1 \bar{P}_1} \frac{d\tilde{N}_1}{d\tilde{t}} - \frac{1}{\tilde{T}_1} \frac{d\tilde{T}_1}{d\tilde{t}}, \tag{3.1b}$$

where we have used (2.1) repeatedly to eliminate $\tilde{N}_j$ from the equation. Here we have divided by fixed (barred) quantities since we need the measurement only once. The flux of moles out of the primary space is the negative of the flux out of the secondary space, so

$$\frac{d\tilde{N}_1}{d\tilde{t}} = -\frac{d\tilde{N}_2}{d\tilde{t}}.$$

Substituting the above into (3.1b), we have

$$\frac{\tilde{p}_2}{\bar{P}_2} - \frac{\tilde{p}_1}{\bar{P}_1} + \frac{1}{\tilde{T}_1} \frac{d\tilde{T}_1}{d\tilde{t}} - \frac{1}{\tilde{T}_2} \frac{d\tilde{T}_2}{d\tilde{t}} = R\frac{d\tilde{N}_2}{d\tilde{t}} \left(\frac{\tilde{T}_2}{V_2 \bar{P}_2} + \frac{\tilde{T}_1}{V_1 \bar{P}_1} \right). \tag{3.2}$$

We now must relate the flux of moles lost to the pressure in the primary space. The leak is sonic (choke) flow if the following identity is satisfied:

$$\frac{\bar{P}_1}{\bar{P}_2} \geq \left(\frac{\gamma + 1}{2} \right)^{\gamma/(\gamma-1)}, \tag{3.3}$$

where $\gamma = c_p/c_v$ is the ratio of the specific heats. To minimize the quantity on the left-hand side, we substitute (A.2a), (A.1), and (A.9) from Appendix A. Thus we have

$$\frac{3.08 \times 10^5 \text{ Pa}}{1.01 \times 10^4 \text{ Pa}} \geq \left(\frac{2.41}{2} \right)^{1.41/0.41}$$

$$3.04 \geq (1.205)^{3.44} = 1.9.$$

Since this bound is always satisfied, we use the following relation between the molar flux and $\tilde{P}_1$, which holds for the sonic case:

$$\frac{d\tilde{N}_2}{d\tilde{t}} = \tilde{a}\tilde{P}_1^{1/2}, \tag{3.4a}$$

$$\tilde{a} = \frac{bA}{M}(\gamma\rho)^{1/2}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(2\gamma-2)}, \tag{3.4b}$$

where M is the molecular weight of the gas, b is a "discharge coefficient", ρ is the density of the gas, and A is the area of the hole. Here the conversion factor $\tilde{a}$ is the only quantity of interest.

Substituting (3.4a) into (3.2) and rearranging, we have

$$\frac{\tilde{p}_2}{\bar{P}_2} - \frac{\tilde{p}_1}{\bar{P}_1} + \frac{1}{\tilde{T}_1}\frac{d\tilde{T}_1}{d\tilde{t}} - \frac{1}{\tilde{T}_2}\frac{d\tilde{T}_2}{d\tilde{t}} = R\tilde{a}\tilde{P}_1^{1/2}\left(\frac{\tilde{T}_2}{V_2\bar{P}_2} + \frac{\tilde{T}_1}{V_1\bar{P}_1}\right) \tag{3.5a}$$

$$\tilde{a} = \frac{1}{R\tilde{P}_1^{1/2}}\left(\frac{\tilde{p}_2}{\bar{P}_2} - \frac{\tilde{p}_1}{\bar{P}_1} + \frac{1}{\tilde{T}_1}\frac{d\tilde{T}_1}{d\tilde{t}} - \frac{1}{\tilde{T}_2}\frac{d\tilde{T}_2}{d\tilde{t}}\right)\left(\frac{\tilde{T}_2}{V_2\bar{P}_2} + \frac{\tilde{T}_1}{V_1\bar{P}_1}\right)^{-1}. \tag{3.5b}$$

Now we further assume that the temperatures are equal across the gap. Then the temperature terms in the first parentheses in (3.5b) cancel, yielding

$$\tilde{a}_{\text{true}} = \frac{1}{R\tilde{P}_1^{1/2}\tilde{T}}\left(\frac{\tilde{p}_2}{\bar{P}_2} - \frac{\tilde{p}_1}{\bar{P}_1}\right)\left(\frac{1}{\bar{P}_2 V_2} + \frac{1}{V_1\bar{P}_1}\right)^{-1}, \tag{3.6}$$

where we have added the subscript "true" to distinguish the mathematically correct value from the experimentally observed value, as described below.

Because each of our measuring devices has an associated measurement error ϵ associated with it, our estimate for $\tilde{a}$ will have an error ϵ_a in it. We approximate this error by linearizing our observed value for $\tilde{a}$ as follows:

$$\tilde{a}_{\text{obs}} = \frac{1}{R[\tilde{P}_1(1-\epsilon_P)]^{1/2}\tilde{T}(1-\epsilon_T)}\left[\frac{\tilde{p}_2(1+\epsilon_p)}{\bar{P}_2(1-\epsilon_P)} - \frac{\tilde{p}_1(1-\epsilon_p)}{\bar{P}_1(1+\epsilon_P)}\right] \times$$
$$\left[\frac{1}{\bar{P}_2(1+\epsilon_P)V_2(1+\epsilon_V)} + \frac{1}{V_1(1+\epsilon_V)\bar{P}_1(1+\epsilon_P)}\right]^{-1}, \tag{3.7}$$

where the subscript "obs" stands for "observed." In (3.7), we note that we have always chosen the sign of the error to make $\tilde{a}_{\text{obs}}$ as large as possible. Performing the linearization, we have

$$\tilde{a}_{\text{obs}} = \frac{(1+\epsilon_P/2)(1+\epsilon_T)(1+\epsilon_P)(1+\epsilon_V)}{R\tilde{P}_1^{1/2}\tilde{T}}\left[\frac{\tilde{p}_2}{\bar{P}_2}(1+\epsilon_p)(1+\epsilon_P) - \frac{\tilde{p}_1}{\bar{P}_1}(1-\epsilon_p)(1-\epsilon_P)\right] \times$$
$$\left(\frac{1}{\bar{P}_2 V_2} + \frac{1}{V_1\bar{P}_1}\right)^{-1}$$

$$= \frac{(1 + 3\epsilon_P/2 + \epsilon_T + \epsilon_V)}{R\tilde{P}_1^{1/2}\tilde{T}} \left[\frac{\tilde{p}_2}{\bar{P}_2} - \frac{\tilde{p}_1}{\bar{P}_1} + (\epsilon_p + \epsilon_P)\left(\frac{\tilde{p}_2}{\bar{P}_2} + \frac{\tilde{p}_1}{\bar{P}_1} \right) \right] \left(\frac{1}{\bar{P}_2 V_2} + \frac{1}{V_1 \bar{P}_1} \right)^{-1}$$

$$= \tilde{a}_{\text{true}} + \epsilon_a$$

$$\epsilon_a = \left(\frac{3\epsilon_P}{2} + \epsilon_T + \epsilon_V \right)\tilde{a}_{\text{true}} + \frac{\epsilon_p + \epsilon_P}{R\tilde{P}_1^{1/2}\tilde{T}}\left(\frac{\tilde{p}_2}{\bar{P}_2} + \frac{\tilde{p}_1}{\bar{P}_1} \right)\left(\frac{1}{\bar{P}_2 V_2} + \frac{1}{V_1 \bar{P}_1} \right)^{-1}. \tag{3.8}$$

Note that the error consists of two parts: a part which is proportional to the measurement (and hence zero if there is no leak), and a part which is absolute.

We then approximate all our variable pressures and temperatures by the baseline values, since the variations are small. To compare the relative contributions to the last term, we calculate the following ratio:

$$\frac{\bar{P}_1 V_1}{\bar{P}_2 V_2} = \frac{(4.46 \times 10^5 \text{ Pa})(100 \text{ gal})}{(1.01 \times 10^5 \text{ Pa})(110 \text{ gal})} = 4.01. \tag{3.9}$$

where we have used (A.2b), (A.1), and (A.4). One might be able to use this result to claim that $\bar{P}_1 V_1 \gg \bar{P}_2 V_2$. Note that this is a tenuous result under the highest-pressure conditions, so it will be less valid under standard operating conditions and not at all appropriate for the vapor recovery line.

Such an assumption would tend to increase the size of the error (since we are reducing the denominator of the second term), and so it would lead to a more conservative estimate. On the other hand, now that (3.8) is in a simple linear form for the error, including the more complicated denominator does not make the underlying analysis more difficult. Thus we work with (3.8) directly, since we do not want an *unnecessarily* conservative error estimate to increase the costs of the equipment needed to control it.

False Positive

Now we wish to examine the case of a false positive reading. In this case, the only pressure differentials are due to temperature variations. Reworking (2.3) in each region and allowing the pressure and temperature to vary, we have

$$\frac{\tilde{p}_j^T}{\tilde{P}_j} = \frac{1}{\tilde{T}}\frac{d\tilde{T}}{d\tilde{t}}. \tag{3.10}$$

In the case of a false reading, $\tilde{a}_{\text{true}} = 0$. Substituting this result and (3.10) into (3.8), we obtain

$$\epsilon_{a,+} = \frac{2(\epsilon_p + \epsilon_P)}{R\tilde{P}_1^{1/2}\tilde{T}^2}\frac{d\tilde{T}}{d\tilde{t}}\left(\frac{1}{\bar{P}_2 V_2} + \frac{1}{V_1 \bar{P}_1} \right)^{-1}, \tag{3.11}$$

where the subscript "+" refers to "false positive".

For a given test, we want to minimize the right-hand side of (3.6). The only parameter that is really at our discretion is $\tilde{P}_1$; the others are pretty much dictated by operating conditions. Thus to minimize the false signal we should increase $\tilde{P}_1$ as much as possible. Thus we choose P_{h} as our test pressure. On the other hand, in order to get the most

conservative estimate for the error, we should use the lower end of the temperature range in (A.6).

Next we look at the error terms. For ϵ_p, we take the ratio of (2.9) to (2.5):

$$\epsilon_p = \frac{9.96 \times 10^{-1} \text{ Pa/hr}}{1.78 \times 10^2 \text{ Pa/hr}} = 5.57 \times 10^{-3}. \tag{3.12}$$

The absolute pressure sensors currently in use have $\epsilon_P = 0.05$. Substituting these results, along with (2.4), (A.1), (A.2b), (A.4), and (A.6) into (3.11), we obtain

$$\epsilon_{a,+} = \frac{2(5.57 \times 10^{-3} + 0.05)}{(4.46 \times 10^5 \text{ Pa})^{1/2}(283 \text{ K})^2} \left(0.5 \frac{\text{K}}{\text{hr}}\right) \left(8.314 \frac{\text{N} \cdot \text{m}}{\text{mol} \cdot \text{K}}\right)^{-1}$$

$$\times \left[\frac{1}{(1.01 \times 10^5 \text{ Pa})(110 \text{ gal})} + \frac{1}{(4.46 \times 10^5 \text{ Pa})(100 \text{ gal})}\right]^{-1}$$

$$= \frac{5.557 \times 10^{-2} \text{ mol}}{(5.35 \times 10^7)(8.314)\text{hr} \cdot \text{Pa}^{3/2} \cdot \text{m}^3} \left(\frac{\text{gal} \cdot \text{Pa}}{1.12 \times 10^{-7}}\right) \left(\frac{3.7853 \times 10^{-3} \text{ m}^3}{\text{gal}}\right)$$

$$= (7.57 \times 10^{-5})(5.557 \times 10^{-2}) \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}} = 4.21 \times 10^{-6} \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}}. \tag{3.13}$$

Since the value in (3.13) is over twice that of the threshold value listed in (A.8), we see that a false positive is possible with $\epsilon_P = 5\%$. Hence we now solve to see what value of ϵ_P will make us get the threshold value with 95% confidence, given that ϵ_P is the 95% confidence level for the meter. Thus we repeat the calculation in (3.13), but replace the left-hand side by $\tilde{a}_c$ and allow ϵ_P to vary:

$$\tilde{a}_c = (7.57 \times 10^{-5})(5.57 \times 10^{-3} + \epsilon_P) \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}}$$

$$1.44 \times 10^{-6} = 4.22 \times 10^{-7} + 7.57 \times 10^{-5}\epsilon_P$$

$$\epsilon_P = \frac{1.02 \times 10^{-6}}{7.57 \times 10^{-5}} = 1.35\% \tag{3.14}$$

However, in order to make sure we don't get any false negative readings, we would probably want to cut this value in half. This would lead to a 1% pressure sensor error, which is commercially available, but expensive.

False Negative

Now we test for a false negative. Suppose that there is a leak. In that case there $\tilde{a}_{\text{true}} \neq 0$ and we must consider both terms in (3.8). First we note that since in the false-positive case, we bounded the second term in (3.8) away from $\tilde{a}_{\text{true}}$, we see that the second term will dominate the first, which is only a fraction of $\tilde{a}_{\text{true}}$.

By switching the sign of the terms with subscript 1 in (3.5a) and using the fact that the temperature is isothermal, we have

$$\frac{\tilde{p}_2}{\bar{P}_2} + \frac{\tilde{p}_1}{\bar{P}_1} - \frac{2}{\tilde{T}} \frac{d\tilde{T}}{d\tilde{t}} = R\tilde{a}_{\text{true}}\tilde{P}_1^{1/2} \left(\frac{\tilde{T}}{V_2\bar{P}_2} - \frac{\tilde{T}}{V_1\bar{P}_1}\right)$$

$$\frac{1}{R\tilde{T}\tilde{P}_1^{1/2}} \left(\frac{\tilde{p}_2}{\bar{P}_2} + \frac{\tilde{p}_1}{\bar{P}_1}\right) = \frac{2}{R\tilde{T}^2\tilde{P}_1^{1/2}} \frac{d\tilde{T}}{d\tilde{t}} + \tilde{a}_{\text{true}} \left(\frac{1}{\bar{P}_2V_2} - \frac{1}{\bar{P}_1V_1}\right).$$

$$\epsilon_a = \left(\frac{3\epsilon_P}{2} + \epsilon_T + \epsilon_V\right)\tilde{a}_{\text{true}} + \epsilon_{a,+} + (\epsilon_p + \epsilon_P)\tilde{a}_{\text{true}}\left(\frac{1}{\bar{P}_2 V_2} - \frac{1}{\bar{P}_1 V_1}\right)\left(\frac{1}{\bar{P}_2 V_2} + \frac{1}{V_1 \bar{P}_1}\right)^{-1}$$

$$= \left[\frac{3\epsilon_P}{2} + \epsilon_T + \epsilon_V + (\epsilon_p + \epsilon_P)\left(\frac{\bar{P}_1 V_1 - \bar{P}_2 V_2}{\bar{P}_2 V_2 + V_1 \bar{P}_1}\right)\right]\tilde{a}_{\text{true}} + \epsilon_{a,+},$$

where we have used (3.10). Then by the arguments given at the beginning of this subsection, we see that the extra contribution to the error from the leak is negligible compared to the original in the false positive error.

In other words, to leading order the error in observation is absolute; that is, there is the same size error no matter whether there is a leak or not. This should help us in designing a tolerance threshold to reduce the possibility of false negatives or false positives. Though the latter is more expensive because of the unnecessary excavation of the site, the former could subject the company to legal difficulties.

Section 4: Small Oscillations

Since we can't use absolute amplitude measurements to determine the leak, we can also attempt to use frequency data. To do so, we introduce a variable flow rate $\tilde{Q}$ into the primary space, using complex arithmetic for simplicity:

$$\tilde{Q}(\tilde{t}) = Q_\Delta e^{i\omega_Q \tilde{t}}, \tag{4.1a}$$

where ω_Q is the frequency associated with the oscillation. In current use, the meters have enough noise in them that it takes a 10-minute rolling average to establish trends. Thus, we should make sure that ω_Q is defined to take that into account. We also neglect any thermal effects this forcing might produce, as well as any pressure fluctuations in the wall.

If we assume that the temperature also oscillates in the same way:

$$\tilde{T}(\tilde{t}) = \bar{T}\left(1 + T_\Delta e^{i\omega_T \tilde{t}}\right), \tag{4.1b}$$

then we would like $\omega_Q \gg \omega_T$ so that the fast flux oscillations can be isolated from the slow temperature oscillations. If we are able to do so, it is possible that we could then introduce a vacuum into the secondary space to amplify the effects of the leak. In general, this is inadvisable since gas could then penetrate into the secondary space from the soil if there were holes in the secondary wall. These leaks would then contaminate simple pressure measurements. On the other hand, by constructing the "high-pass" filter described here, we could screen out those spurious measurements.

Equations (4.1) motivate the following scalings:

$$Q(t) = \frac{\tilde{Q}(\tilde{t})}{Q_\Delta}, \qquad \tilde{T}(\tilde{t}) = \bar{T}\left[1 + T_\Delta T(t)\right], \qquad t = \omega_Q \tilde{t}. \tag{4.2}$$

Here we have chosen to measure time on the scale of the flux oscillations so that the temperature fluctuations seem very slow. Substituting (4.2) into (4.1), we obtain

$$Q(t) = e^{it}, \tag{4.3a}$$

$$T(t) = e^{i\omega_r t}, \qquad \omega_r = \frac{\omega_T}{\omega_Q} \ll 1, \tag{4.3b}$$

where the subscript "r" refers to ratio.

Now we establish a relationship between the moles of gas in each chamber and Q. We begin by considering the secondary space. Substituting (2.1) into (3.4a), we have

$$\frac{d\tilde{N}_2}{d\tilde{t}} = \tilde{a}\left(\frac{\tilde{N}_1 R \tilde{T}_1}{V_1}\right)^{1/2}. \tag{4.4a}$$

Since this deficit must come directly from the primary space, we have that

$$\frac{d\tilde{N}_1}{d\tilde{t}} = Q_\Delta e^{i\omega_Q \tilde{t}} - \tilde{a}\left(\frac{\tilde{N}_1 R\tilde{T}_1}{V_1}\right)^{1/2}, \tag{4.4b}$$

where the extra flux term we are introducing now provides a net addition to the concentration. Equations (4.4) now provide a set of two equations in the unknowns $\tilde{N}_j$. Then once that has been done, $\tilde{P}_2$ can be determined from $\tilde{N}_2$.

In order to scale $\tilde{N}_1$, we simply normalize by the baseline value given current operating conditions. This average is currently unknown, but can be computed for either domain by rearranging (2.1):

$$\bar{N}_j = \frac{\bar{P}_j V_j}{R\bar{T}}. \tag{4.5}$$

We do this because we would like to have the flexibility to handle cases where the oscillations are the order of the original baseline operating condition. Thus we have

$$\tilde{N}_1(\tilde{t}) = \bar{N}_1 N_1(t). \tag{4.6}$$

Substituting (4.6) and (4.2) into (4.4b), we obtain

$$\bar{N}_1 \omega_Q \dot{N}_1 = Q_\Delta e^{it} - \tilde{a}\left(\frac{\bar{N}_1 R\bar{T}_1}{V_1}\right)^{1/2} N_1^{1/2}(1 + T_\Delta e^{i\omega_r t})^{1/2}$$

$$\dot{N}_1 = \frac{Q_\Delta}{\bar{N}_1 \omega_Q} e^{it} - \frac{\tilde{a}\bar{P}_1^{1/2}}{\bar{N}_1 \omega_Q} N_1^{1/2}$$

$$= q e^{it} - a N_1^{1/2}, \tag{4.7}$$

$$q = \frac{Q_\Delta}{\bar{N}_1 \omega_Q} = \frac{\text{flux particles}}{\text{baseline particles}} \tag{4.8a}$$

$$a = \frac{\tilde{a}\bar{P}_1^{1/2}}{\bar{N}_1 \omega_Q} = \frac{\text{leak particles}}{\text{flux particles}}, \tag{4.8b}$$

where the dot indicates differentiation with respect to t. Moreover, we have used the fact (derived in the Appendix) that $T_\Delta \ll 1$ to simplify our expressions. Note also from the physical interpretations of our parameters that we expect both to be small.

In order to scale $\tilde{N}_2$, we use the fact that the operating pressure is going to be much higher than any perturbations we might introduce. Thus, we let

$$\tilde{N}_2(\tilde{t}) = \bar{N}_2[1 + N_{2,\Delta} N_2(t)]. \tag{4.9}$$

Substituting (4.6), (4.9) and (4.2) into (4.4a), we have

$$\bar{N}_2 N_{2,\Delta} \omega_Q \dot{N}_2 = \tilde{a}\left(\frac{\bar{N}_1 R\bar{T}_1}{V_1}\right)^{1/2} N_1^{1/2}(1 + T_\Delta e^{i\omega_r t})^{1/2}$$

$$\dot{N}_2 = \frac{\tilde{a}\bar{P}_1^{1/2}}{\bar{N}_2 N_{2,\Delta} \omega_Q} N_1^{1/2}. \tag{4.10}$$

Thus by letting

$$N_{2,\Delta} = \frac{\tilde{a}\bar{P}_1^{1/2}}{\omega_Q \bar{N}_2},$$

we could end up with a very simple equation. However, we note that if there is no leak, then $\tilde{a} = 0$. Thus we want to leave $\tilde{a}$ in our equation. So instead we choose

$$N_{2,\Delta} = \frac{\bar{N}_1}{\bar{N}_2},$$

from which we have

$$\dot{N}_2 = \frac{\tilde{a}\bar{P}_1^{1/2}}{\bar{N}_1 \omega_Q} N_1^{1/2} = aN_1^{1/2}. \tag{4.11}$$

Note that by retaining this scaling, we retain the balance between $\dot{N}_1$ and $\dot{N}_2$ even in the scaled context.

Section 5: Tracking Temperature Changes

Now we wish to verify that the variation of temperature listed in (2.4) is consistent with temperature variations at the surface. To do this, we use the heat equation

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} = \alpha_s^2 \frac{\partial^2 \tilde{T}}{\partial \tilde{z}^2}, \qquad t \geq 0, \quad z \geq 0, \tag{5.1a}$$

where α_s^2 is the thermal diffusivity of soil (hence the subscript "s") and $\tilde{z}$ measures distance below the surface. At the surface, the temperature is assumed known and of the form in (4.1b):

$$\tilde{T}(0, \tilde{t}) = \bar{T}\left(1 + T_\Delta e^{i\omega_T \tilde{t}}\right). \tag{5.1b}$$

Now that we have a time scale, we may introduce dimensionless similarity variables into our problem. The form of (5.1b) motivates the following separation-of-variables solution:

$$\tilde{T}(\tilde{z}, \tilde{t}) = \bar{T}\left[1 + T_\Delta f(z)e^{i\omega_T \tilde{t}}\right], \qquad z = \frac{\tilde{z}}{\alpha}\sqrt{\omega_T} \tag{5.2}$$

where $f(\tilde{z})$ is to be determined. Substituting (5.2) into (5.1), we obtain

$$i\omega_T T_\Delta f(\tilde{z})e^{i\omega_T \tilde{t}} = \alpha_s^2 T_\Delta \frac{\omega_T}{\alpha_s^2} f''(z)e^{i\omega_T \tilde{t}}$$

$$f'' - if = 0 \tag{5.3a}$$

$$\bar{T}\left[1 + T_\Delta f(0)e^{i\omega_T \tilde{t}}\right] = \bar{T}\left(1 + T_\Delta e^{i\omega_T \tilde{t}}\right)$$

$$f(0) = 1. \tag{5.3b}$$

The solution of (5.3) is

$$f(z) = \exp\left(-z\sqrt{i}\right) = \exp\left(-\frac{z(1+i)}{\sqrt{2}}\right), \tag{5.4}$$

where we have chosen the negative square root since we want the solution to be bounded as $z \to \infty$. Because of the way we have scaled the problem, $|f(z)|$ is the fraction of the surface temperature variation experienced at z. Note that the temperature variation in the soil is both smaller than and phase-shifted from the temperature at the surface. The phase shift in particular might be troubling, though we do have one temperature sensor in the tank already that should allow us to track it.

To actually measure the variation at some depth $\tilde{z}_*$, we note from (5.2) that

$$\frac{\partial \tilde{T}/\partial \tilde{t}(\tilde{z}_*, \tilde{t})}{\partial \tilde{T}/\partial \tilde{t}(0, \tilde{t})} = \frac{i\omega_T \bar{T} T_\Delta f(z_*)e^{i\omega_T \tilde{t}}}{i\omega_T \bar{T} T_\Delta f(0)e^{i\omega_T \tilde{t}}} = f(z_*). \tag{5.5}$$

We are most interested in the change in the temperature at the pipe depth, which we consider to be $\tilde{z}_* = 0.5$ m. With the values in (A.13), we may calculate the dimensionless depth:

$$z_* = (0.5 \text{ m})\sqrt{(7.27 \times 10^{-5} \text{ s}^{-1})\left(5 \times 10^{-7} \frac{\text{m}^2}{\text{s}}\right)^{-1}} = (0.5)\sqrt{145.4} = 6.04. \qquad (5.6)$$

Thus

$$|f(z_*)| = \exp\left(-\frac{6.04}{\sqrt{2}}\right) = e^{-4.27} = 1.41 \times 10^{-2}. \qquad (5.7)$$

In an extreme case, surface temperatures may change by 20 K over the course of a day. Since this is a one-way variation, we would then have that (at a maximum),

$$\left|\frac{\partial \tilde{T}}{\partial \tilde{t}}(0,\tilde{t})\right| = \frac{20 \text{ K}}{12 \text{ hr}}$$

$$\left|\frac{\partial \tilde{T}}{\partial \tilde{t}}(\tilde{z}_*,\tilde{t})\right| = |f(z_*)|\left(1.67 \frac{\text{K}}{\text{hr}}\right) = (1.41 \times 10^{-2})\left(1.67 \frac{\text{K}}{\text{hr}}\right) = 2.34 \times 10^{-2} \frac{\text{K}}{\text{hr}}, \qquad (5.8)$$

where we have used (5.5). This is an order of magnitude smaller than what is discussed in section 2, but over shorter time scales the temperatures may vary by more.

To check the relative size of the variation, we note that our average temperature $\bar{T}$ should be the average of our values in (A.6), since the bounds in (A.6) are achieved during diurnal temperature variations. Thus we take

$$\bar{T} = 285.6 \text{ K}. \qquad (5.9)$$

Since the 20 K/day is an absolute variation, not a variation about a mean, we must halve it to calculate T_Δ:

$$\left|\frac{\partial \tilde{T}}{\partial \tilde{t}}(0,\tilde{t})\right| = \left|i\omega_T \bar{T} T_\Delta e^{i\omega_T \tilde{t}}\right| = \omega_T \bar{T} T_\Delta$$

$$\left(1.67 \frac{\text{K}}{\text{hr}}\right) = 2(7.27 \times 10^{-5} \text{ s}^{-1})(285.6 \text{ K})T_\Delta$$

$$T_\Delta = \frac{1.67}{4.15 \times 10^{-2}} \frac{\text{s}}{\text{hr}}\left(\frac{\text{hr}}{3600 \text{ s}}\right) = 1.11 \times 10^{-2}. \qquad (5.10)$$

Thus, note that the percentage temperature variation, even at the surface, is only around 1% of the total, and the temperature variation at greater depth is even smaller than that (by a factor of $f(z)$).

Next we wish to examine the temperature variation within the pipe itself. A schematic is shown in Fig. 5.1. To make the analysis simpler, we would like to model the temperature
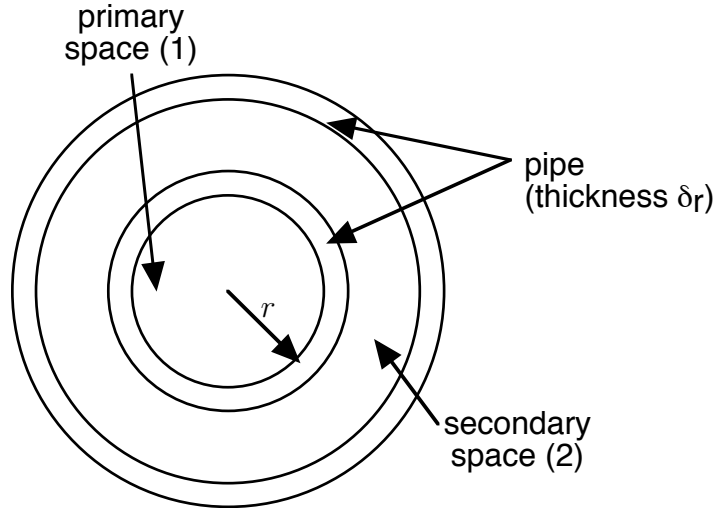
Figure 5.1. Schematic of pipeline.

on the outside of the pipe as in (5.1b). Of course, that neglects the spatial variation described in (5.4). To get a relevant estimate of the size, we compute the following ratio:

$$\frac{\left|\tilde{T}(0.5 - (R_2 + \delta_2), \tilde{t}) - \tilde{T}(0.5 + (R_2 + \delta_2), \tilde{t})\right|}{\bar{T}T_\Delta |f(6.04)|}$$

$$= \frac{\text{spatial variation between top and bottom of pipe at time } \tilde{t}}{\text{amplitude of temporal variations at } \tilde{z}_* = 0.5},$$

where R_2 is the radius of the secondary space (interior of secondary pipe) and δ_2 is the secondary pipe thickness. Using the values in (A.11), we obtain

$$\left|\tilde{T}(0.5 - (R_2 + \delta_2), \tilde{t}) - \tilde{T}(0.5 + (R_2 + \delta_2), \tilde{t})\right|$$

$$= \left|\bar{T}\left[1 + T_\Delta f((0.5 - (4.23 \times 10^{-2} + 2.16 \times 10^{-3}))\sqrt{145.4})e^{i\omega_T \tilde{t}}\right]\right.$$

$$\left. - \bar{T}\left[1 + T_\Delta f((0.5 + (4.23 \times 10^{-2} + 2.16 \times 10^{-3}))\sqrt{145.4})e^{i\omega_T \tilde{t}}\right]\right|$$

$$\frac{|\tilde{T}(0.5 - 0.044, \tilde{t}) - \tilde{T}(0.5 + 0.044, \tilde{t})|}{\bar{T}T_\Delta} = |f(6.04)|\left|\exp\left(\frac{-0.536(1 + i)}{\sqrt{2}}\right)\right.$$

$$\left. - \exp\left(-\frac{0.536(1 + i)}{\sqrt{2}}\right)\right|$$

$$\frac{|\tilde{T}(0.456, \tilde{t}) - \tilde{T}(0.544, \tilde{t})|}{\bar{T}T_\Delta |f(6.04)|} = 2\left|\sinh\left(\frac{0.536(1 + i)}{\sqrt{2}}\right)\right|$$

$$= 2(5.36 \times 10^{-1})(0.284) = 1.07. \qquad (5.11)$$

Since the variations are roughly the same size, taking the temperature on the radius to be a constant is probably not a good approximation. Nevertheless, we note that with T_Δ so

small, the absolute error we make is small. Therefore, we press on with the idea that this analysis will help us understand the true case.

In polar coordinates, the heat transport equation becomes

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} = \frac{\alpha^2}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{r}} \right), \qquad \tilde{t} \geq 0, \tag{5.12}$$

Motivated by (5.1b), we now hypothesize the following scaling:

$$\tilde{T}(\tilde{r}, \tilde{t}) = \bar{T} \left[1 + T_\Delta f(z_*) g(r) e^{i \omega_T \tilde{t}} \right], \qquad r = \frac{\tilde{r} - r_*}{r_{\mathrm{c}}}, \tag{5.13}$$

where r_{c} is the width in each section, and r_* corresponds to the left endpoint of each section. Thus in each section, $r \in [0, 1]$. Note that with this choice of scaling for $\tilde{T}$, $g = 1$ at the soil surface.

Substituting (5.13) into (5.12), we obtain

$$T_\Delta f(z_*) i \omega_T g e^{i \omega_T \tilde{t}} = T_\Delta f(z_*) e^{i \omega_T \tilde{t}} \frac{\alpha^2}{r_{\mathrm{c}}^2} \frac{1}{r_* + r_{\mathrm{c}} r} \frac{d}{dr} \left((r_* + r_{\mathrm{c}} r) \frac{dg}{dr} \right)$$

$$\frac{d}{dr} \left((r_* + r_{\mathrm{c}} r) \frac{dg}{dr} \right) = i \lambda (r_* + r_{\mathrm{c}} r) g, \qquad \lambda = \frac{\omega_T r_{\mathrm{c}}^2}{\alpha^2} \tag{5.14}$$

Technically, we would have to solve (5.14) in four different domains (secondary and primary spaces, two pipe walls) and couple the solutions (which can be expressed in terms of modified Bessel functions) together. However, a leading-order solution can be obtained with significantly less work.

We begin with the primary space, where $r_* = 0$. From the Appendix, we see that $\lambda_1 \ll 1$. Thus to leading order, (5.14) becomes

$$\frac{d(r g')}{dr} = 0 \tag{5.15a}$$

$$g' = \frac{\mathrm{constant}}{r}. \tag{5.15b}$$

Equation (5.15a) is Laplace's equation. Since the temperature cannot diverge, $g' = 0$ in this region, so g is constant. In the pipe, $\lambda_{\mathrm{p},1} \ll 1$, so we again obtain Laplace's equation. By the compatibility condition, the net flux through the boundaries must be zero. But there is no flux into the primary space since $g' = 0$ at that boundary. Thus there is no flux from the secondary space either, so g is constant in the interior pipe.

Similarly, $\lambda_2 \ll 1$, so Laplace's equation holds to leading order in the secondary space. Since there is no flux into the interior pipe, $g' = 0$ everywhere, and the temperature in the secondary space is constant. Repeating the analysis in the outer pipe (where $\lambda_{\mathrm{p},2} \ll 1$), we see that to leading order the temperature is constant everywhere in the system, so $g = 1$ everywhere. Note that this is simply a leading-order solution; there will be corrections of $O(\lambda)$.

We note that these simplifications wouldn't work if we took a spatially-varying temperature in the soil boundary, since this would then introduce nonzero fluxes into the problem. However, since such errors are on the order of the diurnal temporal variations, at least we know the correct order-of-magnitude estimate for the temperature fluctuations in the pipe. (These computations could be verified by temperature sensors.)

There are several possible drawbacks with this approach, though none of them should seriously affect our solution at leading order.

1. Water in the ground would presumably increase the diffusivity of heat through it.
2. The higher pressure in the primary space may slightly affect the thermal diffusivity.
3. Any convection in the air would effectively increase the transport, reducing any temperature differentials.
4. The pipes have been assumed as concentric circles, though this may not be the case.
5. There's no analysis of how long it takes to settle down to this state. For example, if one fills the primary space with air 20K above the secondary space, how long does it take for the temperatures to equalize?

Section 6: Tracking Wall Flexing

If we introduce flux changes as modeled in (4.1a), these will cause pressure changes in the primary space. If the pressure changes are large enough or the walls of the pipe are flexible enough, the pressure changes would cause deformations in the pipe wall. This would then provide another mechanism for pressure changes in the secondary space, changes not accounted for in section 3.
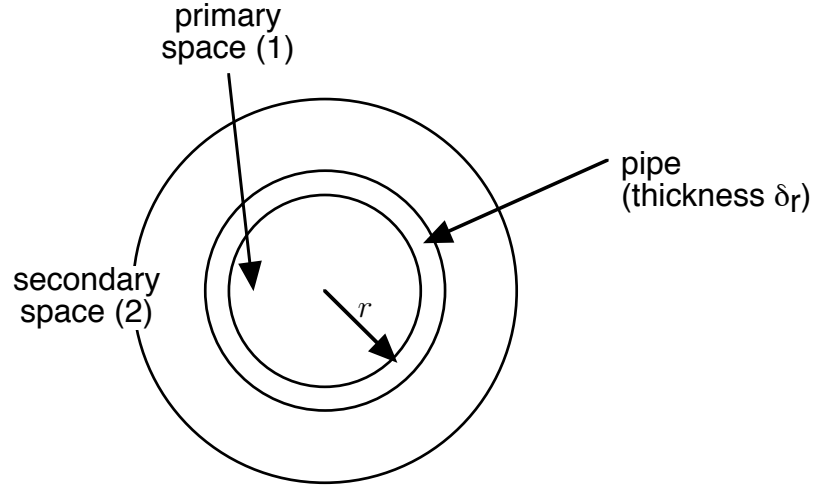

Figure 6.1. Cross-section of spaces (with pipe thickness greatly magnified).

We consider a cross-section of the spaces, as shown in Fig. 6.1. If we consider the cross-section of the pipe to be a hoop, then the relationship between the pressure differential and changes in the radius of the pipe is given by [p. 517 of "Advanced Strength and Applied Elasticity", A.C. Ugural and S.K. Fenster, Prentice-Hall, Upper Saddle River, NJ (1995)]:

$$\sigma = \frac{(\tilde{P}_1 - \tilde{P}_2)R_1}{\delta_1}, \tag{6.1a}$$

where δ_1 is the thickness of the primary pipe and σ is the stress in the hoop (the θ-θ element of the stress tensor in cylindrical coordinates). In order to eliminate σ from (6.1a), we note that the θ-θ element of the strain tensor in cylindrical coordinates is simply the relative change in its radius:

$$\frac{\Delta \tilde{r}_1}{R_1},$$

where $\Delta \tilde{r}_1$ the deviation of the radius of the pipe from its equilibrium value. Thus using the standard stress-strain relationship, we have

$$\sigma = E \frac{\Delta \tilde{r}_1}{R_1}, \tag{6.1b}$$

where E is Young's modulus for the material.

Combining (6.1) and differentiating, we have

$$E\frac{\Delta\tilde{r}_1}{R_1} = \frac{(\tilde{P}_1 - \tilde{P}_2)R_1}{\delta_1}$$

$$\frac{d(\tilde{P}_1 - \tilde{P}_2)}{d\tilde{t}} = \frac{\delta_1 E}{R_1^2}\frac{d(\Delta\tilde{r}_1)}{d\tilde{t}}. \tag{6.2}$$

Since we have only pressure measurements at our disposal, we wish to eliminate the radius change in (6.2). Since the primary space is a cylinder, we have that

$$\frac{d\tilde{V}_1}{d\tilde{t}} = 2\pi R_1 L\frac{d\Delta\tilde{r}_1}{d\tilde{t}}$$

$$\frac{2\pi R_1 L}{\pi R_1^2 L}\frac{d\Delta\tilde{r}_1}{d\tilde{t}} = \frac{1}{\tilde{V}_1}\frac{d\tilde{V}_1}{d\tilde{t}}$$

$$\frac{1}{R_1^2}\frac{d\Delta\tilde{r}_1}{d\tilde{t}} = \frac{1}{2R_1\tilde{V}_1}\frac{d\tilde{V}_1}{d\tilde{t}}, \tag{6.3}$$

where L is the length of the pipe (which we quickly eliminate since we have relatively accurate measurements for the equilibrium volume of the pipe). Substituting (6.3) into (6.2), we have

$$\tilde{p}_2 = \tilde{p}_1 - \frac{\delta_1 E}{2R_1\tilde{V}_1}\frac{d\tilde{V}_1}{d\tilde{t}}. \tag{6.4}$$

We now wish to use (6.4) to estimate $d\tilde{P}_2/d\tilde{t}$ simply due to pressure changes caused by the addition of our flux in (4.4b) with $\tilde{a} = 0$ (since we don't wish to consider the leak at this time). Taking the derivative of (2.1) in the primary space, we have

$$\tilde{p}_1\tilde{V}_1 + \tilde{P}_1\frac{d\tilde{V}_1}{d\tilde{t}} = \frac{d\tilde{N}_1}{d\tilde{t}}(R\tilde{T}_1)$$

$$\tilde{p}_1 = -\frac{\tilde{P}_1}{\tilde{V}_1}\frac{d\tilde{V}_1}{d\tilde{t}} + \frac{R\tilde{T}_1}{\tilde{V}_1}\frac{d\tilde{N}_1}{d\tilde{t}}$$

$$= \tilde{P}_1\left(\frac{1}{\tilde{N}_1}\frac{d\tilde{N}_1}{d\tilde{t}} - \frac{1}{\tilde{V}_1}\frac{d\tilde{V}_1}{d\tilde{t}}\right). \tag{6.5a}$$

Next we consider the secondary space. Since there is no leak, $\tilde{N}_2$ doesn't change. Moreover, if we consider the secondary space to have an infinitely rigid outer wall, any change in $\tilde{V}_1$ due to pressure changes must be balanced by an equal an opposite in $\tilde{V}_2$. Using these facts when taking the derivative of (2.1) in the secondary space, we have

$$\tilde{p}_2\tilde{V}_2 + \tilde{P}_2\frac{d\tilde{V}_2}{d\tilde{t}} = 0$$

$$\frac{d\tilde{V}_1}{d\tilde{t}} = \frac{\tilde{V}_2}{\tilde{P}_2}\tilde{p}_2. \tag{6.5b}$$

Substituting (6.5a) into (6.4) and then using (6.5b), we have

$$\tilde{p}_2^{\mathrm{f}} = \tilde{P}_1 \left(\frac{1}{\tilde{N}_1} \frac{d\tilde{N}_1}{d\tilde{t}} - \frac{1}{\tilde{V}_1} \frac{d\tilde{V}_1}{d\tilde{t}} \right) - \frac{\delta_1 E}{2 R_1 \tilde{V}_1} \frac{d\tilde{V}_1}{d\tilde{t}}$$

$$= \frac{\tilde{P}_1}{\tilde{N}_1} \frac{d\tilde{N}_1}{d\tilde{t}} - \frac{1}{\tilde{V}_1} \left(\tilde{P}_1 + \frac{\delta_1 E}{2 R_1} \right) \left(\frac{\tilde{V}_2}{\tilde{P}_2} \tilde{p}_2 \right)$$

$$= \frac{R\tilde{T}_1}{\tilde{V}_1} \frac{d\tilde{N}_1}{d\tilde{t}} \left[1 + \frac{\tilde{V}_2}{\tilde{V}_1 \tilde{P}_2} \left(\tilde{P}_1 + \frac{\delta_1 E}{2 R_1} \right) \right]^{-1}, \tag{6.6}$$

where we have used the superscript "f" to indicate changes due only to wall flexing.

To get an estimate of the value, we replace most of our functions with the associated constants and use the values from the Appendix for the rigid fiberglass pipe:

$$\bar{p}_2^{\mathrm{f}} = \frac{R\tilde{T}_1}{\tilde{V}_1} \frac{d\tilde{N}_1}{d\tilde{t}} \left[1 + \frac{V_2}{V_1 P_{\mathrm{a}}} \left(P_{\mathrm{o}} + \frac{\delta_1 E}{2 R_1} \right) \right]^{-1}$$

$$= \left(8.314 \, \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{mol} \cdot \mathrm{K}} \right) \left(\frac{288 \, \mathrm{K}}{100 \, \mathrm{gal}} \right) \left(\frac{\mathrm{gal}}{3.7853 \, \mathrm{L}} \right) \left(\frac{\mathrm{L}}{10^{-3} \, \mathrm{m}^3} \right)^{-1} \frac{d\tilde{N}_1}{d\tilde{t}} \times$$

$$\left\{ 1 + \frac{110 \, \mathrm{gal}}{(100 \, \mathrm{gal})(1.01 \times 10^5 \, \mathrm{Pa})} \times \right.$$

$$\left. \left[3.08 \times 10^5 \, \mathrm{Pa} + \frac{(2.03 \times 10^{-3} \, \mathrm{m})(1.2548 \times 10^{10} \, \mathrm{Pa})}{2(2.82 \times 10^{-2} \, \mathrm{m})} \right] \right\}^{-1}$$

$$= \frac{6.33 \times 10^3 \, \mathrm{Pa/mol}}{1 + 1.09 \times 10^{-5}(3.08 \times 10^5 + 4.52 \times 10^8)} \frac{d\tilde{N}_1}{d\tilde{t}}$$

$$= \frac{6.33 \times 10^3 \, \mathrm{Pa/mol}}{4.93 \times 10^3} \frac{d\tilde{N}_1}{d\tilde{t}} = \left(1.29 \, \frac{\mathrm{Pa}}{\mathrm{mol}} \right) \frac{d\tilde{N}_1}{d\tilde{t}}. \tag{6.7}$$

In the above we used the high end of the range in (A.6) to get the maximum effect.

Note that the P_{o} term in the reciprocal is negligible compared to the other term, so (6.7) is an accurate approximation for all $\bar{P}_1$. We now wish to compute the ratio of this term to our leak term. First we note that because of the form of (3.4a), the molar leak rate is proportional to $\bar{P}_1^{1/2}$. Since the secondary space is always assumed to be at the same pressure, this means that the volumetric leak rate is also proportional to $\bar{P}_1^{1/2}$, and hence by (2.7b) we see that $\tilde{p}_2^{\mathrm{l}}$ is as well. Since (2.8) is calculated for P_{o}, we see that for any other operating pressure, we have

$$\bar{p}_2^{\mathrm{l}} = 6.51 \, \frac{\mathrm{Pa}}{\mathrm{hr}} \left(\frac{\bar{P}_1}{P_{\mathrm{o}}} \right)^{1/2}. \tag{6.8}$$

Substituting (6.8) and (4.4b) (with no leak) into (6.7), we obtain

$$\frac{\bar{p}_2^{\mathrm{f}}}{\bar{p}_2^{\mathrm{l}}} = \left(1.29 \, \frac{\mathrm{Pa}}{\mathrm{mol}} \right) \frac{d\tilde{N}_1}{d\tilde{t}} \left[6.51 \, \frac{\mathrm{Pa}}{\mathrm{hr}} \left(\frac{\bar{P}_1}{P_{\mathrm{o}}} \right)^{1/2} \right]^{-1}$$

$$= \frac{(3.08 \times 10^5 \text{ Pa})^{1/2}}{\bar{P}_1^{1/2}} (1.97 \times 10^{-1})(Q_\Delta e^{i\omega\tilde{t}}) \frac{\text{hr}}{\text{mol}}$$

$$\left| \frac{\bar{p}_2^{\text{f}}}{\bar{p}_2^{\text{l}}} \right| = (1.09 \times 10^2) \frac{Q_\Delta}{\bar{P}_1^{1/2}} \frac{\text{hr} \cdot \text{Pa}^{1/2}}{\text{mol}}. \tag{6.9}$$

Since we want this ratio to be small, we have that

$$Q_\Delta \ll \frac{\bar{P}_1^{1/2}}{1.09 \times 10^2} \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}}$$

$$\ll 9.10 \times 10^{-3} \bar{P}_1^{1/2} \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}}. \tag{6.10}$$

The expression in (6.10) provides the estimate for the size of Q_Δ we were lacking in section 4. Note that the upper bound makes physical sense, since we are going to ensure that we don't flex the pipe walls enough to cause difficulties.

In particular, if we use the pressurized pipe, then we have

$$Q_\Delta \ll (9.10 \times 10^{-3})(4.46 \times 10^5 \text{ Pa})^{1/2} \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}} = 6.08 \frac{\text{mol}}{\text{hr}},$$

where we have used (A.2b). This seems like a pretty tight bound.

Note that the analysis here was for the rigid pipe. If E for the flexible pipe is smaller, then the size of P_1 might come into play in (6.7).

Section 7: Using Tracer Gases

Another approach to track leaks is the use of tracer gases . Current tracer gas technology (patented by another compnay) consists of pumping pressurized nitrogen into the primary space and probing the ground above and around the system for escaping gas over a period of two or three days. If there is a leak in the secondary space, nitrogen will be present in the soil. Because of the long time scale needed to produce results, this approach is undesirable.
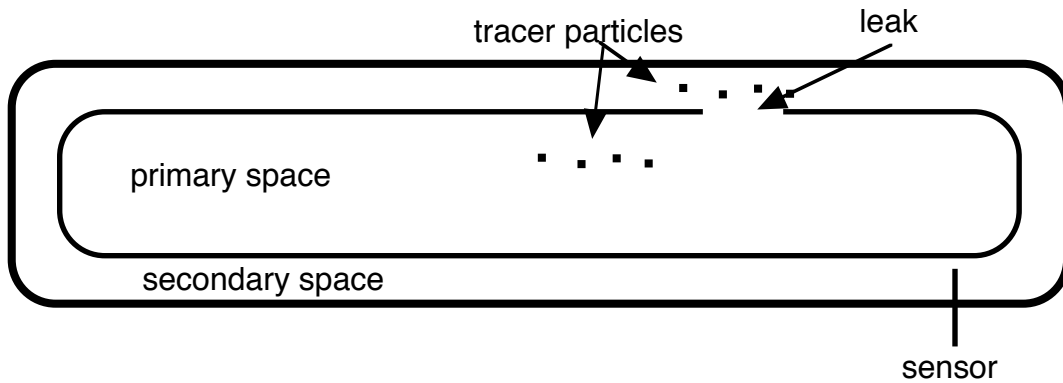

Figure 7.1. Schematic of pipeline with leak (greatly magnified).

We now propose a method by which the tracer-gas approach can be adapted to the Veeder-Root technology. Consider the system in Figure 7.1. Since we can't place detection sensors everywhere in the secondary space, a leak in another area won't be detected on a reasonable time scale because diffusion is too slow.
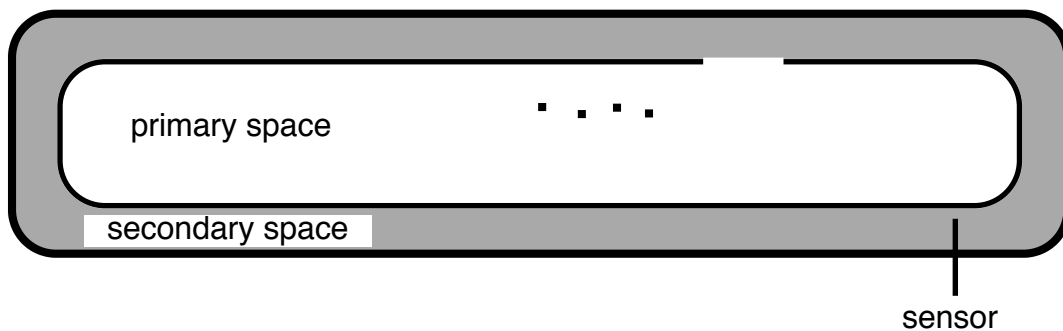

Figure 7.2. Schematic of pipeline with circulated leak.

Therefore, we must introduce a mechanism to shorten the time scale, a mechanism we will call Circulated Vapor Leak Detection (CVLD). Suppose that we can introduce some convection into the secondary space, as shown in Fig. 7.2. This will then equilibrate the number of tracer particles throughout the secondary space, allowing the sensor to detect the leak.

The goal of CVLD is to achieve flow through the secondary space. Assume that the secondary space is constructed so that flow is possible. Note that this process will necessitate the addition of certain equipment to the system in order to produce the convection. However, we note that

1. The valves, etc., needed to hook up pumps are already present on the equipment as part of the pressurization protocol.
2. Such pumps should be relatively low-cost.

To begin CVLD, fill the secondary space with nitrogen at ambient pressure and pump air pressurized to about 64 psi into the primary space. After three to four hours of circulation in the secondary space, we sample a given volume. Such a time estimate assumes standard laminar Poiseuille flow in the annular secondary space. Treating elbows in the pipe or considering turbulent flow is beyond the scope of this project. Nevertheless, there does seem to be relevant research available in the literature (Belanger *et al.*, 1994).

Using an oxygen sensor (which may or may not be permanently installed on-site), we would then sample the test volume for oxygen concentration (in ppm). If it is below some critical threshold, we conclude that there is no leak. Otherwise, the exact ppm count would indicate the size of the leak but not its location. (The location determination will be performed below.)

One thing we must consider is the amount of tracer gas that would be introduced into the secondary space. Would it be detectable using affordable instruments? Let's start by looking at standard operating conditions, so the volumetric leak rate is given by (A.7). Oxygen is 21% of air, so we would have

$$\frac{N(O_2)}{N_2} = \frac{V(O_2)}{V_2} = \left(5 \times 10^{-3} \frac{\text{gal}}{\text{hr}}\right)(0.21)\frac{\tilde{t}}{77.8 \text{ gal}} = \frac{1.25 \times 10^{-5}\tilde{t}}{\text{hr}} = 13 \text{ ppm} \frac{\tilde{t}}{\text{hr}}.$$

In going from the first fraction to the second, we have ignored any additional pressure contribution to P_2 from the additional gas, since it is so small. Hence if we were to run the test for an hour, we should see 13 ppm of oxygen if there is a leak. This level of detection is well within the range of affordable devices. Pressurizing the primary space would raise the threshold even more.
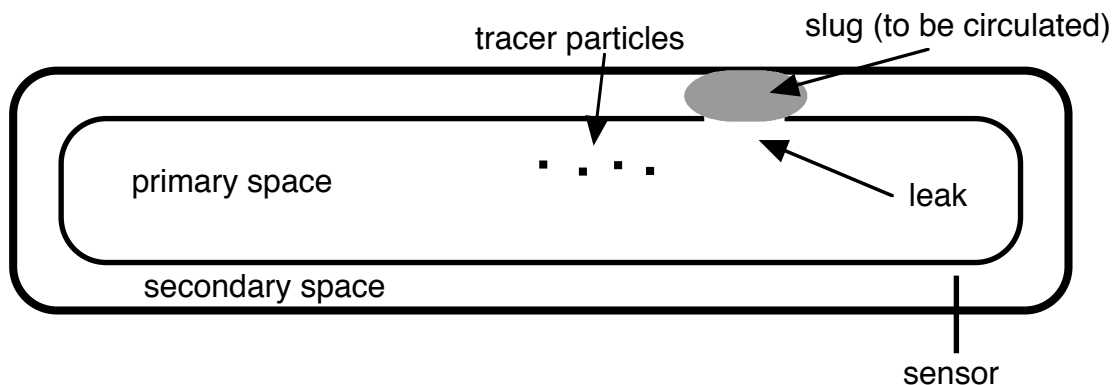

Figure 7.3. Schematic of leak location procedure.

If a leak is detected, we reset the system by voiding the secondary space and sealing it, thus allowing the tracer particles to diffuse and form a slug near the leak (see Fig. 7.3). We then introduce pressure in one direction at a velocity that will cause laminar flow in the annular, secondary space. This will necessitate a balance between two competing forces. In order to guarantee near-laminar flow (especially in the presence of elbows in the pipe, etc.), the pressure head should be very small. On the other hand, the test must be fast enough to be practical.

The time it takes for the slug to reach the sensor under the presence of the pressure head is recorded. We repeat the process (void and seal, wait, introduce pressure, record time slug takes to reach sensor), but this time introduce the pressure from the opposite direction. With these two recorded times, we should be able to estimate the position of the leak.

To conclude this section, we present a mathematical model of simple diffusion of the tracer gas into a semi-infinite one-dimensional medium. Suppose that we fill the secondary space with nitrogen and then use oxygen as a tracer gas. As a first approach, we could model the transport by one-dimensional diffusion only. Doing so, we obtain

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} = D \frac{\partial^2 \tilde{C}}{\partial \tilde{z}^2}, \qquad 0 \le \tilde{z}, \tag{7.1}$$

where $\tilde{z} = 0$ is the position of the leak, D is the diffusion coefficient, and $\tilde{C}$ is the concentration of oxygen, and the secondary space is taken to be semi-infinite.

Initially there is no oxygen in the secondary space, so

$$\tilde{C}(\tilde{z}, 0) = 0. \tag{7.2}$$

Similarly, in the far field there will be no oxygen, so

$$\tilde{C}(\infty, t) = 0. \tag{7.3}$$

Lastly, the flux through the hole is specified—here we choose it to be constant:

$$D \frac{\partial \tilde{C}}{\partial \tilde{z}}(0, \tilde{t}) = -Q. \tag{7.4}$$

Equations (7.1)–(7.4) are amenable to a similarity-variable approach. Thus we let

$$\tilde{C}(\tilde{z}, \tilde{t}) = C_{\mathrm{c}} \tilde{t}^n f(\zeta), \qquad \zeta = \frac{\tilde{z}}{2\sqrt{D\tilde{t}}}, \tag{7.5}$$

where C_{c} and n are unknown constants. Substituting (7.5) into (7.4) and denoting differentiation with respet to ζ with a prime, we have

$$\frac{D}{2\sqrt{D\tilde{t}}} C_{\mathrm{c}} \tilde{t}^n f'(0) = -Q \qquad \Longrightarrow \qquad n = \frac{1}{2} \tag{7.6a}$$

$$\frac{C_{\mathrm{c}}}{2} \sqrt{D} f'(0) = -Q \qquad \Longrightarrow \qquad C_{\mathrm{c}} = \frac{2Q}{\sqrt{D}}. \tag{7.6b}$$

$$f'(0) = -1. \tag{7.7}$$

Substituting (7.5) with the values in (7.6) into (7.1)–(7.3), we have

$$C_{\mathrm{c}}\left(\frac{f}{2\sqrt{\tilde{t}}} - \sqrt{\tilde{t}}\frac{\tilde{z}}{4\tilde{t}^{3/2}}f'\right) = D\frac{C_{\mathrm{c}}}{4D\tilde{t}}\sqrt{\tilde{t}}f''$$

$$2f - 2\zeta f' = f'', \tag{7.8}$$

$$f(\infty) = 0. \tag{7.9}$$

The solution of (7.8) subject to (7.7) and (7.9) can be found as

$$f(\zeta) = \zeta(\operatorname{erf}\zeta - 1) + \frac{2}{\sqrt{\pi}}e^{-\zeta^2}. \tag{7.10}$$
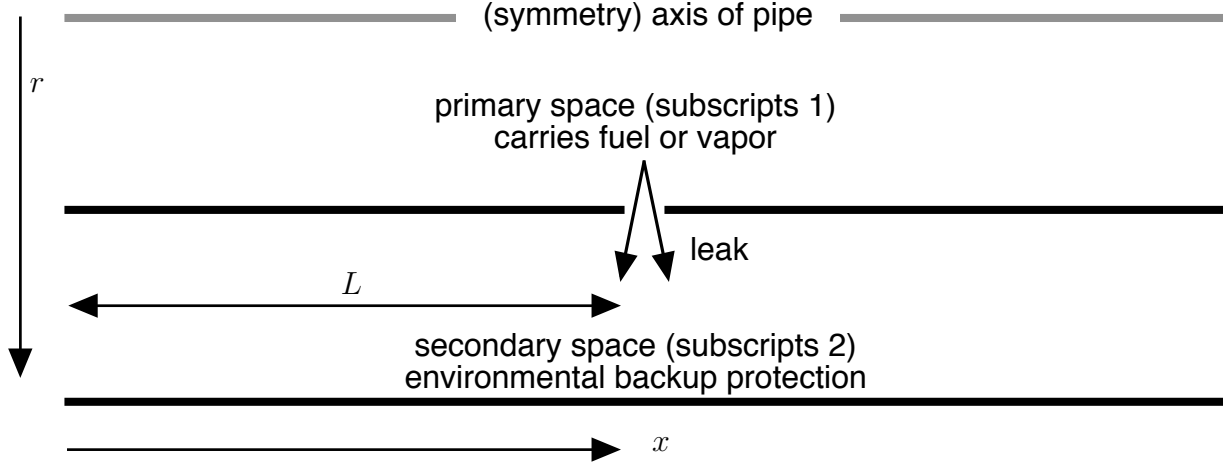
Section 8: Acoustic Detection

Figure 8.1. Schematic of pipeline.

In this section we examine the feasibility of using acoustic waves to determine the position of a leak (the existence of which may have been detected by other means). Consider a leak between the primary chamber and the secondary chamber in a double-walled pipe at a distance L away from one end of the pipe ($\tilde{x} = 0$), as shown in Figure 8.1.

We assume that there are different gases in the two spaces to facilitate the analysis. However, for simplicity, we would like to use an effective density within the second chamber, and relate this to the pressure and temperature through the ideal gas law relationship.

We know that with a mixture of gases in the second chamber, the total pressure can be written as

$$\tilde{P}_2 = R\tilde{T}\frac{\tilde{N}^{(1)}}{V_2} + R\tilde{T}\frac{\tilde{N}^{(2)}}{V_2}, \tag{8.1a}$$

where the superscript refers to the species of gas. (This choice also relates to the inital position of the gas, with species 1 being stored in the primary space and species 2 occupying the secondary space.) In addition, the overall mass density of the gas is given by

$$\tilde{\rho} = \tilde{\rho}^{(1)} + \tilde{\rho}^{(2)}, \qquad \rho^{(j)} = \frac{\tilde{N}^{(j)}M^{(j)}}{V_2}. \tag{8.1b}$$

Combining (8.1), we obtain

$$\tilde{P}_2 = R\tilde{T}\left[\frac{\tilde{\rho}^{(1)}}{M^{(1)}} + \frac{\tilde{\rho}^{(2)}}{M^{(2)}}\right]. \tag{8.2}$$

The pipes considered are meters long, while the gap spacing in the secondary chamber is on the order of centimeters. Thus we assume that the the dynamics within the secondary chamber are one-dimensional. (This assumption would be wrong only in the instance that

the leak is very close to one end of the chamber.) Finally, since the size of the leak is expected to be on the order of microns, so we can represent the leak as a point source of mass at a location $\tilde{x} = L$.

We begin by writing down conservation of mass for the two species (Kevorkian, 1990):

$$\frac{\partial \tilde{\rho}^{(1)}}{\partial \tilde{t}} + \frac{\partial (\tilde{\rho}^{(1)} \tilde{u})}{\partial \tilde{x}} = D \frac{\partial^2 \tilde{\rho}^{(1)}}{\partial \tilde{x}^2} + M^{(1)} \tilde{Q} \delta(\tilde{x} - L), \tag{8.3a}$$

$$\frac{\partial \tilde{\rho}^{(2)}}{\partial \tilde{t}} + \frac{\partial (\tilde{\rho}^{(2)} \tilde{u})}{\partial \tilde{x}} = D \frac{\partial^2 \tilde{\rho}^{(2)}}{\partial \tilde{x}^2}, \tag{8.3b}$$

where $\tilde{u}$ is the velocity in the $\tilde{x}$-direction. Note the introduction of the parameter $M^{(1)}$ to convert the molar leak defined in (4.1a) into a mass leak. Note that the source term comes into play only in (8.3a), since it is species 1 that leaks from the primary space.

Conservation of momentum and energy are given by

$$[\tilde{\rho}^{(1)} + \tilde{\rho}^{(2)}] \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} \right) + \frac{\partial \tilde{P}_2}{\partial \tilde{x}} = \frac{4\mu}{3} \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} \tag{8.4a}$$

$$[\tilde{\rho}^{(1)} + \tilde{\rho}^{(2)}] c_v \left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{T}}{\partial \tilde{x}} \right) + \left(\tilde{P}_2 - \frac{4\mu}{3} \frac{\partial \tilde{u}}{\partial \tilde{x}} \right) \frac{\partial \tilde{u}}{\partial \tilde{x}} = k \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2}, \tag{8.4b}$$

where μ is the dynamic viscosity of the gas and k is the thermal conductivity of the gas.

To scale the dependent variables, we let

$$u(x,t) = \frac{\tilde{u}(\tilde{x}, \tilde{t})}{u_c}, \qquad P_2(x,t) = \frac{\tilde{P}_2(\tilde{x}, \tilde{t})}{P_a}, \qquad \rho^{(j)}(x,t) = \frac{\tilde{\rho}^{(j)}(\tilde{x}, \tilde{t})}{\rho_a}, \qquad T(x,t) = \frac{\tilde{T}(\tilde{x}, \tilde{t})}{\bar{T}}, \tag{8.5a}$$

$$x = \frac{\tilde{x} - L}{x_c}, \qquad t = \frac{\tilde{t} u_c}{x_c}, \tag{8.5b}$$

where u_c and x_c are undetermined for now. To calculate ρ_a, we note that the secondary space is associated with species 2, so we can calculate it from known quantities using (8.2):

$$\rho_a = \frac{P_a M^{(2)}}{R \bar{T}}. \tag{8.6}$$

Note that in principle for a mixture with spatially varying density ρ, the parameters μ, k, and c_v can all vary in space. In the following, we consider gas viscosity to be negligible and let $\mu = 0$.

Substituting (8.5) and (8.6) into (8.2), we obtain

$$P_a P_2 = \rho_a R \bar{T} T \left[\frac{\rho^{(1)}}{M^{(1)}} + \frac{\rho^{(2)}}{M^{(2)}} \right]$$

$$P_2 = M^{(2)} T \left[\frac{\rho^{(1)}}{M^{(1)}} + \frac{\rho^{(2)}}{M^{(2)}} \right]$$

$$= T[M_r \rho^{(1)} + \rho^{(2)}], \qquad M_r = \frac{M^{(2)}}{M^{(1)}}. \tag{8.7}$$

Substituting (8.5) and (8.6) into (8.3a), we obtain

$$\frac{\rho_a u_c}{x_c}\frac{\partial \rho^{(1)}}{\partial t} + \frac{\rho_a u_c}{x_c}\frac{\partial(\rho^{(1)}u)}{\partial x} = \frac{D\rho_a}{x_c^2}\frac{\partial^2 \rho^{(1)}}{\partial x^2} + M^{(1)}\tilde{Q}\delta(x)$$

$$\frac{\partial \rho^{(1)}}{\partial t} + \frac{\partial(\rho^{(1)}u)}{\partial x} = \mathrm{Pe}^{-1}\frac{\partial^2 \rho^{(1)}}{\partial x^2} + Q\delta(x). \tag{8.8a}$$

$$\mathrm{Pe} = \frac{u_c}{D/x_c}, \qquad Q = \frac{M^{(1)}\tilde{Q}x_c}{u_c\rho_a}, \tag{8.8b}$$

where Pe is the Peclet number and Q is the dimensionless leak flow rate. Similarly, (8.3b) becomes

$$\frac{\partial \rho^{(2)}}{\partial t} + \frac{\partial(\rho^{(2)}u)}{\partial x} = \mathrm{Pe}^{-1}\frac{\partial^2 \rho^{(2)}}{\partial x^2}. \tag{8.9}$$

Substituting (8.5) into (8.4a) and using our assumption about the invscid limit, we obtain

$$\rho_a[\rho^{(1)} + \rho^{(2)}]\left(\frac{u_c^2}{x_c}\frac{\partial u}{\partial t} + \frac{u_c^2}{x_c}u\frac{\partial u}{\partial x}\right) + \frac{P_a}{x_c}\frac{\partial P_2}{\partial x} = 0$$

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{P_a}{\rho_a u_c^2}\frac{\partial P_2}{\partial x} = 0$$

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\bar{c}^2/\gamma}{u_c^2}\frac{\partial P_2}{\partial x} = 0, \qquad \bar{c}^2 = \frac{\gamma P_a}{\rho_a}, \tag{8.10a}$$

where $\bar{c}$ is the sound speed in the secondary space if only species 2 occupied it. Continuing to simplify, we have

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{1}{\mathrm{Ma}^2\gamma}\frac{\partial P_2}{\partial x} = 0, \qquad \mathrm{Ma} = \frac{u_c}{\bar{c}}. \tag{8.10b}$$

Here Ma is the Mach number.

Substituting (8.5) into (8.4b) and using the inviscid approximation, we obtain

$$\frac{c_v\rho_a\bar{T}u_c}{x_c}[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) + \frac{u_c}{x_c}P_aP_2\frac{\partial u}{\partial x} = \frac{k\bar{T}}{x_c^2}\frac{\partial^2 T}{\partial x^2}$$

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) + \frac{P_a}{\rho_a\bar{T}c_v}P_2\frac{\partial u}{\partial x} = \frac{D}{x_c u_c}\frac{k}{c_v\rho_a D}\frac{\partial^2 T}{\partial x^2}$$

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) + \frac{R}{M^{(2)}c_v}P_2\frac{\partial u}{\partial x} = \frac{\mathrm{Le}}{\mathrm{Pe}}\frac{\partial^2 T}{\partial x^2}, \qquad \mathrm{Le} = \frac{k}{Dc_v\rho_a}\tag{8.11a}$$

Here Le is the Lewis number. The term in its denominator is often referred to as the thermal diffusivity, and hence the Lewis number is a ratio of transport to thermal effects.

But the definition of the gas constant for species 2 is given by

$$\frac{R}{M^{(2)}} = c_p - c_v,$$

where c_p is the specific heat of the gas mixture at constant pressure. Thus (8.11a) becomes

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) + \frac{c_p - c_v}{c_v}P_2\frac{\partial u}{\partial x} = \frac{\text{Le}}{\text{Pe}}\frac{\partial^2 T}{\partial x^2}$$

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) + (\gamma - 1)P_2\frac{\partial u}{\partial x} = \frac{\text{Le}}{\text{Pe}}\frac{\partial^2 T}{\partial x^2}. \tag{8.11b}$$

Diffusion Limit: Two Species

We first examine the dynamics on the diffusion length and time scale. Thus we set

$$x_c = L_D, \qquad u_c = \frac{D}{L_D} \qquad \Longrightarrow \qquad t_D = \frac{\tilde{t}D}{L_D^2}, \qquad u_D(x_D, t_D) = \frac{L_D\tilde{u}(\tilde{x}, \tilde{t})}{D}, \tag{8.12}$$

where we use the subscript D to indicate diffusion. Here L_D is a diffusion length scale. Hence it is much longer than the leak scale (so the δ-function approximation for the leak still holds) but much less than L.

Because $L_D \ll L$, x can be thought of as an interior-layer variable. Thus we must match to the outer (undisturbed) solution as $x \to \pm\infty$. Thus for the density and temperature, we have

$$\frac{\partial \rho^{(1)}}{\partial x_D}(-\infty, t) = \frac{\partial \rho^{(1)}}{\partial x_D}(\infty, t) = \frac{\partial \rho^{(2)}}{\partial x_D}(-\infty, t) = \frac{\partial \rho^{(2)}}{\partial x_D}(\infty, t) = 0, \tag{8.13a}$$

$$\frac{\partial T}{\partial x_D}(-\infty, t) = \frac{\partial T}{\partial x_D}(\infty, t) = 0. \tag{8.13b}$$

In addition, since the effects of the leak will not be felt upstream, we have that

$$u(-\infty) = 0. \tag{8.14}$$

Substituting (8.12) into (8.8b) and (8.10b), we obtain

$$\text{Pe}_D = \frac{D/L_D}{D/L_D} = 1, \qquad Q_D = \frac{M^{(1)}\tilde{Q}L_D}{(D/L_D)\rho_a} = \frac{M^{(1)}\tilde{Q}L_D^2}{D\rho_a}, \qquad \text{Ma}_D = \frac{D/L_D}{\bar{c}}. \tag{8.15}$$

$\text{Ma}_D \ll 1$ in this case. Equation (3.3) indicates that the leak itself is in the sonic regime, but the length scale is on the order of microns. Moreover, the velocity decreases greatly away from the leak, and hence on the L length scale (which is on the order of centimeters), the flow is subsonic. Hence the leading order of (8.10b) is

$$\frac{\partial P_2}{\partial x_D} = 0, \tag{8.16}$$

so we are in the isobaric case. Note that the momentum equation results in the pressure being a function of time alone, but since we have assumed that the rate Q is constant, then

the pressure drop from the primary to the secondary chambers must also be fixed. Hence for simplicity we treat P_2 as a constant. We admit that this is not a particularly realistic assumption, since there are time-varying temperature-induced changes to P_2 as discussed in section 2. Nevertheless, we expect that the analysis resulting from this assumption will provide insight into the more complicated case.

Then using the fact that P_2 is constant, we may substitute (8.7) and (8.12) into the energy equation (8.11b) to obtain

$$[\rho^{(1)} + \rho^{(2)}]\left(\frac{\partial}{\partial t_D} + u_D\frac{\partial}{\partial x_D}\right)\left(\frac{P_2}{M_r\rho^{(1)} + \rho^{(2)}}\right) + (\gamma - 1)P_2\frac{\partial u_D}{\partial x_D}$$

$$= \mathrm{Le}\frac{\partial^2}{\partial x_D^2}\left(\frac{P_2}{M_r\rho^{(1)} + \rho^{(2)}}\right)$$

$$-\frac{\rho^{(1)} + \rho^{(2)}}{[M_r\rho^{(1)} + \rho^{(2)}]^2}\left(\frac{\partial}{\partial t_D} + u_D\frac{\partial}{\partial x_D}\right)\left[M_r\rho^{(1)} + \rho^{(2)}\right] + (\gamma - 1)\frac{\partial u_D}{\partial x_D}$$

$$= \mathrm{Le}\frac{\partial^2[(M_r\rho^{(1)} + \rho^{(2)})^{-1}]}{\partial x_D^2}$$

$$-\frac{\rho^{(1)} + \rho^{(2)}}{[M_r\rho^{(1)} + \rho^{(2)}]^2}\left\{\frac{\partial^2[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D^2} + M_r Q\delta(x_D) - \frac{\partial\{[M_r\rho^{(1)} + \rho^{(2)}]u_D\}}{\partial x_D}\right.$$

$$\left. +u_D\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D}\right\} + (\gamma - 1)\frac{\partial u_D}{\partial x_D} = \mathrm{Le}\frac{\partial^2[(M_r\rho^{(1)} + \rho^{(2)})^{-1}]}{\partial x_D^2}$$

$$-\frac{\rho^{(1)} + \rho^{(2)}}{[M_r\rho^{(1)} + \rho^{(2)}]^2}\left\{\frac{\partial^2[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D^2} + M_r Q\delta(x_D)\right\}$$

$$+\left[\gamma - 1 + \frac{\rho^{(1)} + \rho^{(2)}}{M_r\rho^{(1)} + \rho^{(2)}}\right]\frac{\partial u_D}{\partial x_D} = \mathrm{Le}\frac{\partial^2[(M_r\rho^{(1)} + \rho^{(2)})^{-1}]}{\partial x_D^2}$$

$$-[\rho^{(1)} + \rho^{(2)}]\left\{\frac{\partial^2[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D^2} + M_r Q\delta(x_D)\right\} + [M_r\rho^{(1)} + \rho^{(2)}]\frac{\partial u_D}{\partial x_D}\times$$

$$\left\{(\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] + \rho^{(1)} + \rho^{(2)}\right\} = \mathrm{Le}[M_r\rho^{(1)} + \rho^{(2)}]^2\frac{\partial^2[(M_r\rho^{(1)} + \rho^{(2)})^{-1}]}{\partial x_D^2},$$

$$(8.17)$$

where in the third line we have used (8.8a) and (8.9).

As with any system involving δ-functions, we solve the homogeneous problem in regions for $x_D < 0$ and $x_D > 0$ and patch them with appropriate jump conditions. Since a second derivative of ρ appears in (8.8a), we immediately have that $\rho^{(j)}$ is continuous across the jump:

$$\rho^{(j)}(0^+, t_D) = \rho^{(j)}(0^-, t_D). \tag{8.18}$$

Then integrating each of (8.8a) and (8.9) across the jump yields

$$\rho^{(j)}(0, t_D)[u_D]_0 = \left[\frac{\partial\rho^{(j)}}{\partial x_D}\right]_0 + \delta_{1j}Q, \tag{8.19a}$$

where the bracket indicates the jump notation. Similarly, integrating (8.17) across the jump yields

$$-[\rho^{(1)} + \rho^{(2)}] \left\{ \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 + M_rQ \right\} + [M_r\rho^{(1)} + \rho^{(2)}][u_D]_0 \times$$

$$\left\{ (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] + \rho^{(1)} + \rho^{(2)} \right\} = \text{Le}[M_r\rho^{(1)} + \rho^{(2)}]^2 \left[\frac{\partial[(M_r\rho^{(1)} + \rho^{(2)})^{-1}]}{\partial x_D} \right]_0$$

$$[\rho^{(1)} + \rho^{(2)}] \left\{ \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 + M_rQ \right\} - [M_r\rho^{(1)} + \rho^{(2)}][u_D]_0 \times$$

$$\left\{ (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] + \rho^{(1)} + \rho^{(2)} \right\} = \text{Le} \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0. \quad (8.19b)$$

Combining (8.19) to obtain the solution, we have

$$[\rho^{(1)} + \rho^{(2)}] \left\{ \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 + M_rQ \right\} - \left\{ \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 + M_rQ \right\} \times$$

$$\left\{ (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] + \rho^{(1)} + \rho^{(2)} \right\} = \text{Le} \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0$$

$$-\left\{ \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 + M_rQ \right\} \left\{ (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] \right\} = \text{Le} \left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0$$

$$-\left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 \left\{ (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] + \text{Le} \right\} = M_rQ \left\{ (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}] \right\}$$

$$\left[\frac{\partial[M_r\rho^{(1)} + \rho^{(2)}]}{\partial x_D} \right]_0 = -M_rQ \left\{ 1 + \frac{\text{Le}}{(\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}]} \right\}^{-1}. \quad (8.20a)$$

$$[M_r\rho^{(1)} + \rho^{(2)}][u_D]_0 = -M_rQ \left\{ 1 + \frac{\text{Le}}{(\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}]} \right\}^{-1} + M_rQ$$

$$= \frac{M_rQ\text{Le}}{(\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}]} \left\{ 1 + \frac{\text{Le}}{(\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}]} \right\}^{-1}$$

$$[u_D]_0 = \frac{M_rQ}{[M_r\rho^{(1)} + \rho^{(2)}]\{1 + (\gamma - 1)[M_r\rho^{(1)} + \rho^{(2)}]/\text{Le}\}}. \quad (8.20b)$$

Diffusion Limit: One Species

Upon further reflection upon the complicated nature of our equations with two species, we note that the same sort of acoustic analysis can be performed even if the species in the primary and secondary space are the same. In that case,

$$M_r = 1, \qquad \rho = \rho^{(1)} + \rho^{(2)}. \quad (8.21)$$

Using (8.21), we see that the ideal gas law (8.7) becomes

$$P_2 = \rho T. \tag{8.22}$$

In addition, equations (8.8a) and (8.9) reduce to a single equation:

$$\frac{\partial \rho}{\partial t_D} + \frac{\partial (\rho u_D)}{\partial x_D} = \frac{\partial^2 \rho}{\partial x_D^2} + Q\delta(x_D), \quad Q = \frac{\tilde{Q}L_D^2}{D}, \tag{8.23}$$

while (8.17) becomes

$$-\rho \left[\frac{\partial^2 \rho}{\partial x_D^2} + Q\delta(x_D) \right] + \rho \frac{\partial u_D}{\partial x_D}[(\gamma - 1)\rho + \rho] = \rho^2 \mathrm{Le} \frac{\partial^2 (\rho^{-1})}{\partial x_D^2}$$

$$- \left[\frac{\partial^2 \rho}{\partial x_D^2} + Q\delta(x_D) \right] + \gamma\rho \frac{\partial u_D}{\partial x_D} = \rho \mathrm{Le} \frac{\partial^2 (\rho^{-1})}{\partial x_D^2}. \tag{8.24}$$

Moreover, (8.18) holds with the superscript removed, and (8.19a) becomes

$$\rho(0, t_D)[u_D]_0 = \left[\frac{\partial \rho}{\partial x_D} \right]_0 + Q. \tag{8.25a}$$

Equation (8.19b) simplifies to

$$\rho \left\{ \left[\frac{\partial \rho}{\partial x_D} \right]_0 + Q \right\} - \rho[u_D]_0[(\gamma - 1)\rho + \rho] = \mathrm{Le} \left[\frac{\partial \rho}{\partial x_D} \right]_0$$

$$\rho \left\{ \left[\frac{\partial \rho}{\partial x_D} \right]_0 + Q - \gamma\rho[u_D]_0 \right\} = \mathrm{Le} \left[\frac{\partial \rho}{\partial x_D} \right]_0, \tag{8.25b}$$

while (8.20) simplifies to

$$\left[\frac{\partial \rho}{\partial x_D} \right]_0 = -Q \left\{ 1 + \frac{\mathrm{Le}}{(\gamma - 1)\rho} \right\}^{-1}, \tag{8.26a}$$

$$[u_D]_0 = \frac{Q}{\rho[1 + (\gamma - 1)\rho/\mathrm{Le}]}. \tag{8.26b}$$

Acoustic Limit: One Species

In order to examine the effect of the change in density upon an acoustic wave entering the pipe, it is preferable to scale velocities by the sound speed $\bar{c}$. In addition, we work on the acoustic time scale $L/\bar{c}$. (Note that the waves we are going to track will move over the entire length of the pipe, so we use L, not L_D.) Thus we have the following to replace (8.12):

$$x_\mathrm{c} = L \quad \Longrightarrow \quad x_c = \frac{\tilde{x}}{L} = \frac{x_D}{L_r}, \quad L_\mathrm{r} = \frac{L}{L_D}, \tag{8.27a}$$

$$u_\mathrm{c} = \bar{c} \quad \Longrightarrow \quad t_c = \frac{\tilde{t}\bar{c}}{L} = \frac{t_D L_D^2}{D L \bar{c}} = \frac{t_D}{\mathrm{Ma}_D L_\mathrm{r}} \tag{8.27b}$$

$$u_c(x_c, t_c) = \frac{\tilde{u}(\tilde{x}, \tilde{t})}{\bar{c}} = \frac{u_D(x, t_D)D}{L_D \bar{c}} = \mathrm{Ma}_D u_D(x_D, t_D), \tag{8.27c}.$$

where we have used (8.15). Here we use the subscript c to emphasize that this is a different time scale from the diffusive limit. From the Appendix, we have that $\text{Ma}_D \ll 1$, so the velocities are much smaller, and $\text{Ma}_D L_\text{r} \ll 1$, so the acoustic time scale is much shorter.

Substituting (8.27) into (8.8b) and (8.10b) and reducing to the one-species case, we obtain

$$\text{Pe}_c = \frac{\bar{c}}{D/L} = \frac{L}{L_D \text{Ma}_D} = \frac{L_\text{r}}{\text{Ma}_D}, \quad Q_c = \frac{M^{(1)} \tilde{Q} L}{\bar{c} \rho_\text{a}} \tag{8.28a}$$

$$\frac{\partial u_c}{\partial t_c} + \left(u_c \frac{\partial u_c}{\partial x_\text{c}} \right) + \frac{1}{\gamma \rho} \frac{\partial (\rho T)}{\partial x_\text{c}} = 0, \tag{8.28b}$$

where we have used (8.22). Because $\bar{c}$ is so large, we have that $\text{Pe} \to \infty$ and $Q \to 0$. In these limits, the leading order of (8.8a) and (8.11b) become (in the one-species case)

$$\frac{\partial \rho}{\partial t_c} + \frac{\partial (\rho u_c)}{\partial x_\text{c}} = 0, \tag{8.30}$$

$$\rho \left(\frac{\partial T}{\partial t_c} + u_c \frac{\partial T}{\partial x_\text{c}} \right) + (\gamma - 1)(\rho T) \frac{\partial u_c}{\partial x_\text{c}} = 0, \tag{8.30a}$$

where we have used (8.22). Equations (8.29), along with (8.28b), form the set of equations that we must solve. Note that since the acoustic time scale is so much shorter than the diffusive time scale, no diffusive terms appear in the equations.

Because the diffusion time scale is so much longer than the acoustic time scale, any solution of the diffusion problem for some time t_* can be considered as a steady-state solution of the acoustic problem. The same argument can also be made for the length scale. Thus we can expand as follows (considering temperature as a representative dependent variable)

$$T(x_D, t_D) = T(x_c L_\text{r}, \text{Ma}_D L_\text{r} t_c) = T_\infty + O(\text{Ma}_D L_\text{r}, L_{|rmr}^{-1}), \tag{8.31}$$

where T_∞ is the value of T at either $x \to \infty$ or $x \to -\infty$. (If these are two distinct values, then the solution will have to be patched up in the boundary layer region where $x = x_D$.)

We want to linearize the problem, so we let

$$T(x_c, t_c) = T_\infty + T_\delta(x_c, t_c) + O(\text{Ma}_D L_\text{r}, L_{|rmr}^{-1}), \tag{8.32a}$$

$$u_D(x_c, t_c) = u_\delta(x_c, t_c) + O(\text{Ma}_D), \tag{8.32b}$$

$$\rho(x_c, t_c) = \rho_\infty + \rho_\delta(x_c, t_c) + O(\text{Ma}_D L_\text{r}, L_{|rmr}^{-1}), \tag{8.32c}$$

where the subscript δ indicates a small perturbation from the initial condition. Note from the form of (8.32) that the perturbations are taken to be of larger size than any of the small parameters in the problem. This ensures that we retain only the linear terms. Note also from (8.32b) that due to the change in our scaling u_c, the leading-order solution for the velocity profile is zero.

Substituting (8.32) into (8.28b) and (8.29), we obtain the following equations, to leading order:

$$\frac{\partial u_\delta}{\partial t} + \frac{1}{\gamma \rho_\infty} \frac{\partial (T_\infty \rho_\delta + T_\delta \rho_\infty)}{\partial x} = 0, \tag{8.33a}$$

$$\frac{\partial \rho_\delta}{\partial t} + \frac{\partial (\rho_\infty u_\delta)}{\partial x} = 0, \tag{8.33b}$$

$$\rho_\infty \frac{\partial T_\delta}{\partial t_c} + (\gamma - 1)(\rho_\infty T_\infty) \frac{\partial u_\delta}{\partial x_c} = 0. \tag{8.33c}$$

If we are considering this to be a signaling problem, (8.33) needs to be solved with $\rho_\delta \to 0, T_\delta \to 0$ and $u_\delta \to e^{i\omega t}$ as $x_c \to -\infty$. Since the energy equation is parabolic, we also need that $T_\delta \to 0$ as $x_c \to \infty$. This is a standard signalling problem for a nonuniform density and temperature profile. Note that for the case of no leak, the density and temperature are constant on this time scale and the signalling problem results in the standard wave equation.

Section 9: Conclusions and Further Research

Pressure measurement. The analysis in Sections 2 and 5 above makes clear that the critical challenge in establishing an effective leak detection system lies in the fact that pressure variations due to diurnal changes in temperature are more than an order of magnitude greater than those due even to an artificially amplified leak in the product line. This renders a simple pressure measurement utterly ineffective. The technique proposed in Section 3 to use measurements in both spaces to cancel the effect of temperature is considerably more promising, but is limited by the fact that the temperature fluctuations still amplify the *error* in the gauges. This error cannot be controlled effectively, and a calculation based on the parameters listed in the appendix shows that gauges with a 5% error in absolute pressure would give rise to a substantial risk of "false positive" leak detections. Our recommendation is that this direct measurement technique only be used with gauges that have at most 1% absolute error.

Forced response. A potentially viable resolution to the issue of signal-to-noise ratio in the pressure measurements is to drive the primary space pressure at a moderate frequency and measure the response in the secondary space. The application of a high-pass filter with appropriate cut-off frequency could then be used to screen out the spurious response due to diurnal fluctuations in temperature. A preliminary consideration of the scalings is given in Section 4, but the idea itself has yet to be analyzed. Section 6 demonstrates that concerns over flexure in the pipe walls masking the forced leak response are not relevant for forcing amplitudes up to 5 mol/hr, or 122 L/hr (compared to a threshold leak rate of 18.9 mL/hr and a total secondary space volume of no greater than 379 L). Based on the promise of this preliminary analysis, we recommend further study of this proposed procedure, starting with what frequency would be an appropriate cut-off for the filtered signal and what the expected damping and lag would be in the measured response.

Direct detection of tracer gas. The use of tracer gases avoids the problems associated with measuring state variables of a single gaseous species. As demonstrated in Section 7, threshold leak rates would produce approximately 13 ppm of oxygen each hour in the secondary space, if the product line's primary space were filled with air and the secondary space initially filled with nitrogen, for instance. This quantity of oxygen is readily detectable with inexpensive commercial gauges, assuming relatively uniform distribution of the oxygen. Section 7 begins an analysis of the time necessary for the tracer to diffuse sufficiently; this analysis has yet to be completed. Also suggested in Section 7 is a method to *locate* the leak after one has been detected. The method proposed would see a pressure head created in the secondary space drive the leaked gas through to a detector in such a way that it remained localized, thus allowing its location to be determined through its time to detection. This process would be repeated with an oppositely charged pressure head to bracket the location of the leak. We recommend that this proposal be analyzed for feasibility, where the system to be considered would be Poiseuille flow with inter-species diffusion, i.e., a 2-D radial extension of the 1-D convection-diffusion model presented in Section 8.

Acoustic detection of tracer gas. A possible alternative for locating the site of the leak exploits a difference in densities of the tracer gas and the gas occupying the secondary space. An acoustic wave sent through the secondary space would scatter off of the nonuniformity in density, and would then be detectable either through its reflected component or through its transmitted component. An analysis of the scalings in a 1-D model, included in Section 7, shows that the disparity between the (slow) diffusion time of the gas emitted into the secondary space via the leak and the characteristic (fast) travel time of an acoustic wave across the leak allows a multiple-scale analysis that effectively breaks the problem into two simpler problems. We recommend that this analysis be completed to determine appropriate operating parameters for optimal detection of the leak site.

Appendix A: Parameter Values

Here are the values of the relevant parameters. For $\bar{P}_2$, there was some discussion of forcing a vacuum in the secondary space. However, that might then cause leaks from the soil into the secondary space. Therefore, we use atmospheric pressure for $\tilde{P}_2$:

$$P_{\mathrm{a}} = (14.696 \text{ psi}) \left(\frac{6.8947 \times 10^3 \text{ Pa}}{\text{psi}} \right) = 1.01 \times 10^5 \text{ Pa}. \tag{A.1}$$

For $\bar{P}_1$, there are two possibilities. The normal operating pressure of the fuel product piping is given by

$$P_{\mathrm{o}} = 30 \text{ psig} = (44.696 \text{ psi}) \left(\frac{6.8947 \times 10^3 \text{ Pa}}{\text{psi}} \right) = 3.08 \times 10^5 \text{ Pa}. \tag{A.2a}$$

(Note that to convert from psig to psi, you simply add atmospheric pressure.) If necessary, we can pressurize the line at a higher pressure P_{h}:

$$P_{\mathrm{h}} = 50 \text{ psig} = (64.696 \text{ psi}) \left(\frac{6.8947 \times 10^3 \text{ Pa}}{\text{psi}} \right) = 4.46 \times 10^5 \text{ Pa}. \tag{A.2b}$$

Note that for some applications the choice of $\bar{P}_1$ doesn't matter, because we are in the choke flow regime whenever the bound in (3.3) is satisfied.

The volume of the primary space is bounded within the following range:

$$2 \text{ gal} \leq V_1 \leq 200 \text{ gal}$$
$$(2 \text{ gal}) \left(\frac{3.7853 \text{ L}}{\text{gal}} \right) \left(\frac{10^{-3} \text{ m}^3}{\text{L}} \right) \leq V_1 \leq (200 \text{ gal}) \left(\frac{3.7853 \text{ L}}{\text{gal}} \right) \left(\frac{10^{-3} \text{ m}^3}{\text{L}} \right)$$
$$7.57 \times 10^{-3} \text{ m}^3 \leq V_1 \leq 7.57 \times 10^{-1} \text{ m}^3. \tag{A.3a}$$

Similarly, the value of the secondary space is

$$1 \text{ gal} \leq V_2 \leq 100 \text{ gal}$$
$$3.79 \times 10^{-3} \text{ m}^3 \leq V_2 \leq 3.79 \times 10^{-1} \text{ m}^3. \tag{A.3b}$$

Since pressure and volume are inversely related in the ideal gas law, for a conservative estimate we always choose values at the upper end of the range so that the pressure variations we have to detect are at the lower end of their range. However, when we want to use actual values, we take

$$V_1 = 100 \text{ gal}, \qquad V_2 = 110 \text{ gal}. \tag{A.4}$$

These values correspond to a 498-ft length of pipe with Ameron Dualoy 3000/L FRP "3-inch over 2-inch" pipe (Ameron, undated). The real dimensions are listed below.

As a first guess, we could surmise that the range of surface temperatures possible would be (FHA, 2002)

$$-10° \text{ C} \leq \tilde{T} \leq 30° \text{ C}. \tag{A.5}$$

However, as shown in section 5, the temperature variation at the pipe wall will be much smaller. Thus we choose the following range as our nominal temperatures for the pipe wall:

$$10° \text{ C} \leq \tilde{T} \leq 15° \text{ C}$$
$$283 \text{ K} \leq \tilde{T} \leq 288 \text{ K}. \tag{A.6}$$

The given leak tolerance we should be able to detect is

$$\frac{d\tilde{V}_l}{d\tilde{t}} = 5 \times 10^{-3} \frac{\text{gal}}{\text{hr}} = \left(5 \times 10^{-3} \frac{\text{gal}}{\text{hr}}\right)\left(\frac{3.7853 \times 10^{-3} \text{ m}^3}{\text{gal}}\right) = 1.89 \times 10^{-5} \frac{\text{m}^3}{\text{hr}}. \tag{A.7}$$

This tolerance is at standard operating conditions, which correspond to the pressure values in (A.2a) and (A.1). In order to compute the loss at varying conditions, we would need to know the dimensions of the leak hole, which are built into $\tilde{a}$ *via* (3.4b). Thus it is most important to convert this into the $\tilde{a}$ value. Combining (2.7a) and (3.4a), we have

$$P_a \frac{d\tilde{V}_l}{d\tilde{t}} = R\bar{T}_2 \tilde{a}_c \tilde{P}_1^{1/2}$$

$$(1.01 \times 10^5 \text{ Pa})\left(1.89 \times 10^{-5} \frac{\text{m}^3}{\text{hr}}\right) = \left(8.314 \frac{\text{N} \cdot \text{m}}{\text{mol} \cdot \text{K}}\right)(288 \text{ K})\tilde{a}(3.08 \times 10^5 \text{ Pa})^{1/2}$$

$$\tilde{a}_c = \left(1.91 \frac{\text{N} \cdot \text{m}}{\text{hr}}\right)\left(1.33 \times 10^6 \frac{\text{N} \cdot \text{m} \cdot \text{Pa}^{1/2}}{\text{mol}}\right)^{-1}$$

$$= 1.44 \times 10^{-6} \frac{\text{mol}}{\text{hr} \cdot \text{Pa}^{1/2}}, \tag{A.8}$$

where we have used (A.2a) and the maximum value of $\bar{T}$ so that we have the smallest possible bound. Here the subscript "c" denotes "critical".

We use the value of γ given for air in Batchelor (1967):

$$\gamma = 1.41. \tag{A.9}$$

To compute the molar loss rate, we substitute our values into (2.7a):

$$(1.01 \times 10^5 \text{ Pa})\left(1.89 \times 10^{-5} \frac{\text{m}^3}{\text{hr}}\right) = \left(8.314 \frac{\text{N} \cdot \text{m}}{\text{mol} \cdot \text{K}}\right)(288 \text{ K})\frac{d\tilde{N}_2}{d\tilde{t}}$$

$$\frac{d\tilde{N}_2}{d\tilde{t}} = \left(1.91 \frac{\text{N} \cdot \text{m}}{\text{hr}}\right)\left(2.39 \times 10^3 \frac{\text{N} \cdot \text{m}}{\text{mol}}\right)^{-1}$$

$$= 7.97 \times 10^{-4} \frac{\text{mol}}{\text{hr}}, \tag{A.10}$$

which is slightly smaller than the value obtained using the STP conversion factor.

For the rigid fiberglass pipe, we have the following values (Ameron, undated):

$$\delta_1 = 0.08 \text{ in} \left(\frac{2.54 \text{ cm}}{\text{in}} \right) = 2.03 \times 10^{-3} \text{ m},$$

$$\delta_2 = 0.085 \text{ in} \left(\frac{2.54 \text{ cm}}{\text{in}} \right) = 2.16 \times 10^{-3} \text{ m},$$

$$R_1 = 1.11 \text{ in} \left(\frac{2.54 \text{ cm}}{\text{in}} \right) = 2.82 \times 10^{-2} \text{ m}$$

$$R_2 = 1.665 \text{ in} \left(\frac{2.54 \text{ cm}}{\text{in}} \right) = 4.23 \times 10^{-2} \text{ m}.$$

(A.11)

In addition, we have that the Young's modulus is given by

$$E = 12548 \text{ MPa} = 1.2548 \times 10^{10} \text{ Pa}. \tag{A.12}$$

The value for E is the axial modulus of elasticity for $2''$ diameter pipe at $24°$ C, as given by Smith (2004). In order to calculate our temperature variations, we need the following parameters:

$$\alpha_s^2 = 5 \times 10^{-7} \frac{\text{m}^2}{\text{s}}, \qquad \omega_T = \frac{2\pi}{24 \text{ hr}} \left(\frac{\text{hr}}{3600 \text{ s}} \right) = 7.27 \times 10^{-5} \text{ s}^{-1}. \tag{A.13}$$

The value of α_s was estimated as an average for gravel from values of granite and soil in Rolle (1999).

In order to calculate λ, we use the following values for the thermal diffusivity in gas (Rolle, 1999) and in the pipe (Armatec):

$$\alpha_g^2 = 2.2 \times 10^{-5} \frac{\text{m}^2}{\text{s}}, \qquad \alpha_p^2 = 5 \times 10^{-7} \frac{\text{m}^2}{\text{s}}. \tag{A.14}$$

We now (A.14) and (A.13) into our definition of λ in (5.14). For the primary space, $r_c = R_1$. For the secondary space, $r_c = R_2 - (R_1 + \delta_1)$ (the interior spacing). For the pipes, $r_c = \delta$. Using these values, we have

$$\lambda_1 = (2.82 \times 10^{-2} \text{ m})^2 (7.27 \times 10^{-5} \text{ s}^{-1}) \left(2.2 \times 10^{-5} \frac{\text{m}^2}{\text{s}} \right)^{-1}$$

$$= (7.95 \times 10^{-4} \text{ m}^2)(3.3 \text{ m}^{-2}) = 2.63 \times 10^{-3} \ll 1, \tag{A.15a}$$

$$\lambda_2 = [4.23 \times 10^{-2} \text{ m} - (2.82 \times 10^{-2} + 2.03 \times 10^{-3}) \text{ m}]^2 (3.3 \text{ m}^{-2})$$

$$= (1.21 \times 10^{-2} \text{ m})^2 (3.3 \text{ m}^{-2}) = 4.81 \times 10^{-4} \ll 1, \tag{A.15b}$$

$$\lambda_{p,1} = (2.03 \times 10^{-3} \text{ m})^2 (7.27 \times 10^{-5} \text{ s}^{-1}) \left(5 \times 10^{-7} \frac{\text{m}^2}{\text{s}} \right)^{-1}$$

$$= (4.12 \times 10^{-6} \text{ m}^2)(145 \text{ m}^{-2}) = 6.01 \times 10^{-4} \ll 1, \tag{A.15c}$$

$$\lambda_{p,2} = (2.16 \times 10^{-3} \text{ m})^2 (145 \text{ m}^{-2}) = 6.78 \times 10^{-4} \ll 1, \tag{A.15d}$$

To calculate ρ_a, we again consider nitrogen, in which case $M^{(2)}$ becomes (Mahan, 1987):

$$M^{(2)} = \frac{14.01\,\text{g}}{\text{mol}}. \tag{A.16}$$

Substituting (A.1), (A.16), and the lower value in (A.6) into (8.6), we obtain

$$\rho_a = \frac{P_a M^{(2)}}{R\bar{T}} = \frac{1.01 \times 10^5\,\text{Pa}}{283\,\text{K}}\left(\frac{14.01\,\text{g}}{\text{mol}}\right)\left(8.314\,\frac{\text{N} \cdot \text{m}}{\text{mol} \cdot \text{K}}\right)^{-1} = 6.01 \times 10^2\,\frac{\text{g}}{\text{m}^3}$$

$$= 6.03 \times 10^{-1}\,\frac{\text{kg}}{\text{m}^3}. \tag{A.17}$$

Then substituting (A.1), (A.9), and (A.17) into (8.10a), we have that

$$\bar{c}^2 = \frac{\gamma P_a}{\rho_a} = (1.41)(1.01 \times 10^5\,\text{Pa})\left(6.03 \times 10^{-1}\,\frac{\text{kg}}{\text{m}^3}\right)^{-1} = 2.37 \times 10^5\,\frac{\text{m}^3}{\text{m} \cdot \text{s}^2}$$

$$= 2.37 \times 10^5\,\frac{\text{m}^2}{\text{s}^2}. \tag{A.18}$$

In order to calculate the Mach number in the diffusive regime, we need to have at least an order-of-magnitude estimate for D. Taking the value from a CO_2-N_2 gas pair at an appropriate temperature from Bird *et al.* (1960), we have

$$D = 0.158\,\frac{\text{cm}^2}{\text{s}} = 1.58 \times 10^{-5}\,\frac{\text{m}^2}{\text{s}}. \tag{A.19}$$

For the value of L_D, we arbitrarily take

$$L_D = 1\,\text{cm} = 10^{-2}\,\text{m}. \tag{A.20}$$

Note that this is much larger than the leak size and much less than a typical value of L, as required.

Then substituting (A.18)–(A.20) into (8.15), we obtain

$$\text{Ma}_D = \frac{D}{\bar{c}L_D} = \left(1.58 \times 10^{-5}\,\frac{\text{m}^2}{\text{s}}\right)\left(\frac{1}{10^{-2}\,\text{m}}\right)\left(2.37 \times 10^5\,\frac{\text{m}^2}{\text{s}^2}\right)^{-1/2} = 3.25 \times 10^{-6}, \tag{A.21}$$

which is small, as required.

Using the L value discussed above:

$$L = 498\,\text{ft}\left(\frac{12\,\text{in}}{\text{ft}}\right)\left(\frac{\text{m}}{39.37\,\text{in}}\right) = 1.52 \times 10^2\,\text{m}, \tag{A.22}$$

we can then calculate the value of the relevant parameters for the acoustic calculation:

$$L_r = \frac{1.52 \times 10^2\,\text{m}}{10^{-2}\,\text{m}} = 1.52 \times 10^4, \tag{A.23a}$$

$$\text{Ma}_D L_r = (3.25 \times 10^{-6})(1.52 \times 10^4) = 4.93 \times 10^{-2}, \tag{A.23b}$$

which is small, as required.

Appendix B: Verifying the Ideal Gas Law

It is well established that the van der Waals state equation

$$\left[\tilde{P} + b_1 \frac{\tilde{N}^2}{V^2} \right] (V - b_2 \tilde{N}) = \tilde{N} R \tilde{T} \tag{B.1}$$

more accurately reflects physical reality than does the ideal gas state equation. So one needs to determine whether or not this added accuracy is important in the present setting. The short calculation below shows that it is not.

We use the values for $\tilde{P}$ from (A.2a), V as the V_1 value from (A.4), and $\tilde{T}$ as the lower end of (A.6). Assuming that nitrogen is the gas in use, the values for b_1 and b_2 are

$$b_1 = 0.137 \, \frac{\text{J} \cdot \text{m}^3}{\text{mol}^2}, \qquad b_2 = 3.87 \times 10^{-5} \, \frac{\text{m}^3}{\text{mol}}.$$

Substituting these values into the van der Waals equation (B.1) then implies that $\tilde{N} = 49.5$ moles. Differentiating the van der Waals equation (B.1), one finds that

$$\frac{d\tilde{P}}{d\tilde{t}} = \left[\frac{R\tilde{T}}{V - b_2 \tilde{N}} + \frac{b_2 \tilde{N} R \tilde{T}}{(V - b_2 \tilde{N})^2} - 2b_1 \frac{\tilde{N}}{V^2} \right] \frac{d\tilde{N}}{d\tilde{t}} \tag{B.2}$$

Since $b_2 \tilde{N} = 1.92 \times 10^{-3} \, \text{m}^3$ is much smaller than V, one finds that

$$\frac{R\tilde{T}}{V - b_2 \tilde{N}} = 6.22 \times 10^4, \qquad \frac{b_2 \tilde{N} R \tilde{T}}{(V - b_2 \tilde{N})^2} = 31.0, \qquad \frac{2b_1 \tilde{N}}{V^2} = 94.9.$$

This implies that nothing is gained by including a van der Waals correction.

Nomenclature

Units are listed in terms of length (L), mass (M), moles (N), time (T), or temperature (Θ). If the same letter appears both with and without tildes, the letter with a tilde has dimensions, while the letter without a tilde is dimensionless. The equation number where a particular quantity first appears is listed, if appropriate.

A: area of the leak hole, L^2 (3.4b).

$\tilde{a}$: parameter characterizing leak size, $NL^{1/2}/M^{1/2}$ (3.4a).

b: arbitrary parameter, variously defined.

$\tilde{C}(\tilde{z},\tilde{t})$: concentration, N/L^3 (7.1).

c: specific heat, $L^2/T^2\Theta$.

$\bar{c}$: speed of sound for gas, L/T (8.10a).

D: molecular diffusivity, L^2/T (7.1).

E: Young's modulus, M/LT^2 (6.1b).

f: unknown function, variously defined (5.2).

g: unknown function, variously defined (5.13).

j: indexing variable (3.1a).

k: thermal conductivity $ML/T^3\Theta$ (8.4b).

Le: Lewis number (8.11a).

M: molecular weight of gas, M/N (3.4b).

Ma: Mach number (8.10b).

n: arbitrary parameter, variously defined.

$\tilde{N}(\cdot)$: number of moles of gas, N (2.1).

$\tilde{P}(\cdot)$: pressure, M/LT^2 (2.1).

Pe: Peclet number (8.8b).

$\tilde{p}(\cdot)$: $d\tilde{P}/d\tilde{t}$, M/LT^3 (2.1).

$\tilde{Q}(\tilde{t})$: flow rate into primary space, M/T (4.1a).

R: without a subscript, the gas constant, $ML^2/NT^2\Theta$ (2.1); with a subscript, inner radius of pipe, L.

$\tilde{T}(\tilde{t})$: temperature, Θ (2.1).

$\tilde{t}$: time, T (2.4).

$\tilde{u}$: velocity in the acoustic analysis, L/T (8.3a).

V: volume of space, L^3 (2.1).

$\tilde{x}$: distance variable in the acoustic analysis, L (8.3a).

$\tilde{z}$: distance variable, variously defined, L (5.1a).

α^2: thermal diffusivity, L^2/T (5.1a).

δ: pipe thickness, L.

γ: ratio of specific heats, (3.3).

ϵ: error in measurement (3.7).

ζ: similarity variable (7.5).

λ: dimensionless constant (5.14).
ρ: density of gas, N/L^3 (3.4b).
μ: viscosity ML/T (8.4a).
σ: hoop stress, M/LT^2 (6.1a).
ω: frequency of oscillations, T^{-1} (4.1a).
$\mathcal{Z}$: the integers.

Other Notation

a: as a subscript on ϵ, used to indicate the error in $\tilde{a}$.
a: as a subscript, used to indicate atmospheric pressure (2.7a).
c: as a subscript, used to indicate the sound speed scale (8.27).
c: as a subscript, used to indicate a characteristic scale (3.14).
D: as a subscript, used to indicate a diffusion scale (8.5b).
f: as a superscript, used to indicate wall flexing (6.6).
h: as a subscript, used to indicate high pressure (A.2b).
l: as a sub- or superscript, used to indicate the leak (2.7a).
$n \in \mathcal{Z}$: as a subscript, used to indicate the primary or secondary space (2.2a); as a superscript, used to indicate a species of gas (8.1a).
o: as a subscript, used to indicate operating conditions (6.7).
obs: as a subscript, used to indicate the observed value of a quantity (3.7).
P: as a subscript on ϵ, used to indicate the error in $\tilde{P}$ (3.7).
p: as a subscript on ϵ, used to indicate the error in $\tilde{p}$ (3.7).
p: as a subscript, used to indicate the pipe.
Q: as a subscript, used to indicate the flux (4.1a).
r: as a subscript, used to indicate a ratio (4.3b).
s: as a subscript, used to indicate the soil (5.1a).
T: as a sub- or superscript, used to indicate the temperature (2.2b).
true: as a subscript, used to indicate the true (as opposed to observed) value of a quantity (3.6).
V: as a subscript on ϵ, used to indicate the error in V (3.7).
Δ: as a subscript, used to indicate a deviation (4.1a).
$*$: as a subscript, used to indicate a specific value of a quantity.
$+$: as a subscript, used to indicate a false positive (3.11).
$^-$: used to indicate the average or characteristic value of a quantity (2.3).
$[\cdot]_0$: used to indicate the jump in a quantity across $x = 0$ (8.19a).

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